Conformal group of transformations of the quantum field operators in the momentum space and the five dimensional Lagrangian approach

A. I. Machavariani[◇*]
[◇] *Joint Institute for Nuclear Research, Moscow Region*
141980 Dubna, Russia
[*] *High Energy Physics Institute of Tbilisi State University, University Str. 9*
380086 Tbilisi, Georgia

Abstract

Conformal group of transformations in the momentum space, consisting of translations $p'_\mu = p_\mu + h_\mu$, rotations $p'_\mu = \Lambda^\nu_\mu p_\nu$, dilatation $p'_\mu = \lambda p_\mu$ and inversion $p'_\mu = -M^2 p_\mu/p^2$ of the four-momentum p_μ, is used for the five dimensional generalization of the equations of motion for the interacting massive particles. It is shown, that the $\mathcal{S}$-matrix of the charged and the neutral particles scattering is invariant under translations in a four-dimensional momentum space $p'_\mu = p_\mu + h_\mu$. In the suggested system of equations of motion, the one-dimensional equations over the fifth coordinate x_5 are separated and these one dimensional equations have the form of the evaluation equations with $x_5 = \sqrt{x_o^2 - \mathbf{x}^2}$.

The important property of the derived five dimensional equations of motion is the explicit separation of the parts of these equations according to the inversion $p'_\mu = -M^2 p_\mu/p^2$, where M is a scale constant.

In the framework of the considered formulation the five dimensional generalization of the nonlinear σ-model is obtained. The appropriate five dimensional Lagrangian coincides with the usual nonlinear σ-model in the region $p^\mu p_\mu > M^2$. The scale parameter M is determined by the pion mass $2M^2 = m_\pi^2$. Unlike the usual nonlinear σ model, in the proposed Lagrangian the chiral symmetry breaking term $\lambda\sigma$ is exactly reproduced. For the Lagrangian with the spontaneous broken $SU(2)\times U(1)$ symmetry the scale parameter M^2 is determined by the Higgs particle mass $8m_{Higgs}^2 = 9M^2$. In addition, the scalar particle interaction Lagrangian has different signs in the regions $p^\mu p_\mu > M^2$ and $0 < p^\mu p_\mu < M^2$.

Content

Introduction

Conformal transformations of the field operators and corresponding equation of motion in the momentum space were considered in ref. [1, 2, 3, 4]. In these papers conformal transformations were performed in the coordinate space and corresponding relations were constructed in the momentum space using the Fourier transform. Therefore in this approach the translation operator coincides with the four momentum $\widehat{P}_\mu = p_\mu$ in the momentum space. Two particular features determine the advantages of the conformal transformations in the momentum space[3]. First, the real observables of the particle interactions, like the cross sections and the corresponding scattering amplitudes, are determined in the momentum space. Secondly, the accuracy of the measurement of the particle coordinates is in principle restricted by the Compton length of this particle. Moreover in the conformal invariant case the determination of the coordinates of the massless particle generates an additional essential trouble (see [5] ch. 20 and [6]).

In the present paper we consider conformal transformations of the off shell four momentum q_μ ($q_o \neq \sqrt{\mathbf{q}^2 + m^2}$). In this case the generator of the translation in the four-momentum space $\widehat{P}_\mu = i\partial/\partial q_\mu$ coincides with the corresponding coordinate $\widehat{P}_\mu = x_\mu$ in the coordinate space. Translations, rotations, scale transformation and inversion of four momentum q_μ forms the conformal group because the metric tensor of this space transforms under these transformations as $g'_{\mu\nu}(q') = (1 + f(q))g_{\mu\nu}(q)$.

Conformal transformations and their numerous applications are presented in many books and review papers (see for instance [7, 8, 9, 10, 11, 12, 13, 14]). The 6D representation of conformal transformations was done in the Dirac geometrical model [15, 16, 12], where each of the conformal transformations was single-valued reproduced via the appropriate 6D rotation in the invariance 6D cone $\kappa_A\kappa^A \equiv \kappa_\mu\kappa^\mu + \kappa_5^2 - \kappa_6^2 = 0$, where the four momentum q_μ ($\mu = 0, 1, 2, 3$) is defined as $q_\mu = M\kappa_\mu/(\kappa_5 + \kappa_6)$ and M is a scale parameter. The invariance of the 6D cone $\kappa_A\kappa^A = 0$ is valid for each of the conformal transformation even when the conformal invariance is violated by the mass or other dimensional parameters of the interacting particle. Therefore we use this invariance of 6D cone as a constraint for the derivation of the equation of motion for arbitrary interacting massive field. In particular, we will project $\kappa_A\kappa^A = 0$ on the two 5D surface $q_\mu q^\mu + q_5^2 = M^2$ and $q_\mu q^\mu - q_5^2 = -M^2$, so that these 5D hyperboloids are also invariant under the conformal transformations. This invariance enables us to introduce the constraints $\left(q_\mu q^\mu \pm q_5^2 \mp M^2\right)\Phi_\pm(q, q_5) = 0$ for the 5D field operators. Afterwards we construct corresponding 5D Lagrangians and the 5D equation of motion and consider their 4D reductions.

Invariance of quantum field theory under the four-momentum translation $q'_\mu = q_\mu + h_\mu$ in the homogeneous 4D momentum space is interpreted as invariance under a gauge transformation of the charged field $\Phi'_\gamma(x) = \Phi^h_\gamma(x) = e^{ih_\mu x^\mu}\Phi_\gamma(x)$. We will show, that for a neutral field the four-momentum translation has the more complicated form $\Phi'_\gamma(x) = \Phi^h_\gamma(x)$ which does not change the creation or annihilation operator of the considered particle. Thereby the scattering $\mathcal{S}$-matrix is invariant under translations in the four-

dimensional momentum space $q'_\mu = q_\mu + h_\mu$. Moreover, one can generalize invariance of the 6D form $\kappa_A\kappa^A = 0$ and these 5D projections under the translations $q'_\mu = q_\mu + h_\mu$ for other kinds gauge transformations $q'_\mu = q_\mu - eA_\mu(q)$. This enables us to extend suggested 5D approach for the models with the gauge transformations.

The 5D version of the quantum field theory with the invariance form $q_\mu q^\mu + q_5^2 = M^2$ or $q_\mu q^\mu - q_5^2 = -M^2$ was suggested in refs. [17, 18, 19], where the scale parameter M was interpreted as the fundamental (maximal) mass and its inverse $1/M$ as the fundamental (minimal) length [20, 21]. In the present formulation M has the meaning of a boundary parameter which may be determined in the theories with a spontaneous symmetry breaking. In particular, for the nonlinear σ model M is fixed via the mass of π meson and for the standard model with the spontaneous $SU(2) \times U(1)$ symmetry breaking the scale parameter M is determined by the mass of the Higgs particle.

1. Conformal transformations in the 4D momentum space and the scattering $\mathcal{S}$-matrix

Conformal transformations of a four-momentum q_μ ($\mu = 0, 1, 2, 3$) compounded of the following transformations:
translations

$$T(h): \qquad q_\mu \longrightarrow q'_\mu = q_\mu + h_\mu, \tag{1.1a}$$

rotations

$$R(\Lambda): \qquad q_\mu \longrightarrow q'_\mu = \Lambda^\nu_\mu q_\nu, \tag{1.1b}$$

dilatation

$$\mathcal{D}(\lambda): \qquad q_\mu \longrightarrow q'_\mu = e^\lambda \ q_\mu, \tag{1.1c}$$

and inversions

$$I(M^2): \qquad q_\mu \longrightarrow q'_\mu = -M^2 q_\mu / q^2, \tag{1.1d}$$

where a scale parameter M insures the correct dimension of q'_μ. Translation $T(\hbar)$ and inversions $I(M^2)$ form the special conformal transformations

$$\mathcal{K}(M^2, \hbar) \equiv I(M^2)T(\hbar)I(M^2): \qquad q_\mu \longrightarrow q'_\mu = \frac{q_\mu - \hbar_\mu q^2/M^2}{1 - 2q_\nu\hbar^\nu/M^2 + \hbar^2 q^2/M^4}. \tag{1.1e}$$

Obviously, q_μ in (1.1a)-(1.1e) is off mass shell ($q_o \neq \sqrt{\mathbf{q}^2 + m^2}$). Hereafter the on mass shell 4D momenta will be denoted as p_μ ($p^2 = m^2$, $p_o \geq 0$).

Following the Dirac geometrical model [15], transformations (1.1a)-(1.1e) may by realized as rotations in the 6D space with the metric $g_{AB} = diag(+1, -1, -1, -1, +1, -1)$ and on the 6D cone

$$\kappa^2 \equiv \kappa_A\kappa^A = \kappa_\mu\kappa^\mu + \kappa_5^2 - \kappa_6^2 = 0, \tag{1.2}$$

where according to conformal covariant formulation [16, 22],

$$q_\mu = \frac{\kappa_\mu}{\kappa_+}; \quad \kappa_+ = (\kappa_5 + \kappa_6)/M; \quad \mu = 0,1,2,3; \tag{1.3}$$

where κ_+ is a dimensionless scale parameter and κ_μ, q_μ and M have the same dimensions in the system of units $\hbar = c = 1$.

Conformal transformations (1.1a)-(1.1e) of a particle field operator $\Phi_\gamma(x)$:

The particle field operator $\Phi_\gamma(x)$ with a spin-isospin quantum numbers γ is usually expanded in the positive and in the negative frequency parts in the 3D Fock space

$$\Phi_\gamma(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}}\Big[a_{\mathbf{p}\gamma}(x_0)e^{-ipx} + b^+{}_{\mathbf{p}\gamma}(x_0)e^{ipx}\Big]; \quad p_o \equiv \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}, \tag{1.4a}$$

where in the asymptotic regions $a_{\mathbf{p}\gamma}(x_0)$ and $b^+{}_{\mathbf{p}\gamma}(x_0)$ transforms into particle (antiparticle) annihilation (creation) operators $lim_{x_0\to\pm\infty} < m|a_{\mathbf{p}\gamma}(x_0)|n > = < m|a_{\mathbf{p}\gamma}(out,in)|n >$; $lim_{x_0\to\pm\infty} < m|b^+{}_{\mathbf{p}\gamma}(x_0)|n > = < m|b^+{}_{\mathbf{p}\gamma}(out,in)|n >$, where n, m are arbitrary asymptotic states. On the other hand, $\Phi_\gamma(x)$ may be expanded in the 4D momentum space

$$\Phi_\gamma(x) = \int \frac{d^4q}{(2\pi)^4}\Big[\Phi^{(+)}_\gamma(q)e^{-iqx} + \Phi^{(-)^+}_\gamma(q)e^{iqx}\Big]. \tag{1.4b}$$

After comparison of the expressions (1.4a) and (1.4b) we get

$$\frac{e^{-i\omega_{\mathbf{p}}x_o}}{2\omega_{\mathbf{p}}}a_{\mathbf{p}\gamma}(x_o) = \int \frac{dq_o}{2\pi}\Phi^{(+)}_\gamma(q_o,\mathbf{p})e^{-iq_ox_o} \tag{1.5a}$$

$$= i\frac{e^{-i\omega_{\mathbf{p}}x_o}}{2\omega_{\mathbf{p}}}\sum_\beta \int d^3x < 0|\Phi_\beta(x)|\mathbf{p}\gamma > \Big[\frac{\partial\Phi_\beta(x)}{\partial x_o} - i\omega_{\mathbf{p}}\Phi_\beta(x)\Big] \tag{1.5b}$$

and

$$\frac{e^{i\omega_{\mathbf{p_a}}x_o}}{2\omega_{\mathbf{p_a}}}b^+{}_{\mathbf{p_a}\gamma}(x_0) = \int \frac{dq_o}{2\pi}\Phi^{(-)^+}_\gamma(q_o,\mathbf{p_a})e^{iq_ox_o}. \tag{1.6a}$$

$$= -i\frac{e^{i\omega_{\mathbf{p_a}}x_o}}{2\omega_{\mathbf{p_a}}}\sum_\beta \int d^3x < \mathbf{p}_a\gamma|\Phi_\beta(x)|0 > \Big[\frac{\partial\Phi_\beta(x)}{\partial x_o} + i\omega_{\mathbf{p_a}}\Phi_\beta(x)\Big] \tag{1.6b}$$

where we have used the following expressions for a one-particle (antiparticle) states

$$< 0|\Phi_\beta(x)|\mathbf{p}\gamma > = Z^{-1/2}\int \frac{d^3p'}{(2\pi)^3 2\omega_{\mathbf{p}'}}e^{ip'x} < 0|a_{\mathbf{p}'\beta}(x_o)|\mathbf{p}\gamma > = Z^{-1/2}\delta_{\gamma\beta}e^{ipx}, \tag{1.7a}$$

$$< \mathbf{p}_a\gamma|\Phi_\beta(x)|0 > = Z_a^{-1/2}\int \frac{d^3p'}{(2\pi)^3 2\omega_{\mathbf{p}'}}e^{ip'x} < \mathbf{p}_a\gamma|b^+{}_{\mathbf{p}'\gamma}(x_0)|0 > = Z_a^{-1/2}\delta_{\gamma\beta}e^{-ip_ax}, \tag{1.7b}$$

where the index a denotes the antiparticle state and Z is the renormalization constant.

The field operators $a_{\mathbf{p}\gamma}(x_0)$ and $b^{+}{}_{\mathbf{p}\gamma}(x_o)$ are simply defined via the corresponding source operator $\partial/\partial x_0 a_{\mathbf{p}\beta}(x_0) = i \int d^3x e^{ipx} j_\beta(x)$, where $\Big(\partial^2/\partial x_\mu \partial x^\mu + m^2\Big)\Phi_\beta(x) = j_\beta(x)$. Moreover, these operators determine the transition $\mathcal{S}$-matrix

$$\mathcal{S}_{mn} \equiv < out; \mathbf{p}'{}_1\alpha'{}_1, ..., \mathbf{p}'{}_m\alpha'{}_m|\mathbf{p}_1\alpha_1, ..., \mathbf{p}_n\alpha_n; in >= \prod_{i=1}^{m}\Big[\int dx^{0'}{}_i \frac{d}{dx^0{}_i}\Big]$$

$$\prod_{j=1}^{n}\Big[\int dx^0{}_j \frac{d}{dx^0{}_j}\Big] < 0|T\Big(a_{\mathbf{p}'{}_m\alpha'_m}(x^{0'}{}_m), ..., a_{\mathbf{p}'{}_1\alpha'_1}(x^{0'}{}_1)a^{+}_{\mathbf{p}_n\alpha_n}(x^0{}_n), ..., a^{+}_{\mathbf{p}_1\alpha_1}(x^0{}_1)\Big)|0 > . \quad (1.8)$$

Next we will consider the transformations of $\Phi_\beta(x)$ according to the conformal transformations of $\Phi^{(\pm)}_\gamma(q)$

$$\Phi^{(\pm)}_\gamma(q) \to \Phi^{(\pm)'}_\gamma(q') = \mathcal{U}(g)\Phi^{(\pm)}_\gamma(q)\mathcal{U}^{-1}(g) = \mathcal{T}^{\beta}_{\gamma}\Phi^{(\pm)}_\beta(g^{-1}q), \quad (1.9)$$

where g indicates one of the (1.1a)-(1.1e) transformations $g \equiv \Big(T(h), R(\Lambda), \mathcal{D}(\lambda), \mathcal{K}(M, h)\Big)$, $\mathcal{T}^{\beta}_{\gamma}$ is the spin-isopin matrix and $\mathcal{U}(g)$ are defined through the generators of the corresponding transformations in the well known form:

$$T(h): \qquad \mathcal{U}(h) = e^{ih_\mu \mathcal{X}^\mu}; \qquad \Big[\mathcal{X}_\mu, \Phi^{(\pm)}_\gamma(q)\Big] = -i\frac{\partial}{\partial q^\mu}\Phi^{(\pm)}_\gamma(q), \quad (1.10a)$$

$$R(\Lambda): \qquad \mathcal{U}(\Lambda) = e^{i\Lambda_{\mu\nu}\mathcal{M}^{\mu\nu}}; \quad \Big[\mathcal{M}_{\mu\nu}, \Phi^{(\pm)}_\gamma(q)\Big] = -i\Big(q_\mu\frac{\partial}{\partial q^\nu} - q_\nu\frac{\partial}{\partial q^\mu} - i\Sigma_{\mu\nu}\Big)\Phi^{(\pm)}(q) \quad (1.10b)$$

where $\Sigma_{\mu\nu} = 0$ for scalars, $\Sigma_{\mu\nu} = i/4[\gamma_\mu, \gamma_\nu]$ for fermions and $(\Sigma_{\mu\nu}V_\rho) = ig_{\mu\rho}V_\nu - ig_{\nu\rho}V_\mu$ for the vectors V_ρ.

$$\mathcal{D}(\lambda): \qquad \mathcal{U}(\lambda) = e^{i\lambda D}; \qquad \Big[D, \Phi^{(\pm)}_\gamma(q)\Big] = -i\Big(q_\mu\frac{\partial}{\partial q^\mu} + id_m\Big)\Phi^{(\pm)}_\gamma(q), \quad (1.10c)$$

where d_m indicates the scale dimension of field. For example, in the scale-invariant case $d_m = -3$.

$$\mathcal{K}(M, \hbar): \qquad \mathcal{U}(\hbar) = e^{i\hbar_\mu K^\mu}; \qquad \Big[K_\mu, \Phi^{(\pm)}_\gamma(q)\Big] = -\Big(2q_\mu D - q^2\mathcal{X}_\mu + 2iq^\nu\Sigma_{\mu\nu}\Big)\Phi^{(\pm)}_\gamma(q). \quad (1.10d)$$

According to (1.4b) the conformal transformations of the operators $\Phi^{(\pm)}_\gamma(q)$ (1.9) generates the corresponding transformations of $\Phi'_\gamma(x)$

$$\Phi'_\gamma(x) = \mathcal{T}^{\beta}_{\gamma}\int\frac{d^4q}{(2\pi)^4}\Big[\Phi^{(+)}_\beta(g^{-1}q)e^{-iqx} + \Phi^{(-)^{+}}_\beta(g^{-1}q)e^{iqx}\Big]. \quad (1.11)$$

In particular, eq.(1.11) consists of the following transformations:

Four-momentum translation:

For a charged particle a four-momentum translation is equivalent to

$$q'_\mu = q_\mu + h_\mu \qquad \Longrightarrow i\frac{\partial}{\partial x'_\mu} = i\frac{\partial}{\partial x_\mu} + h_\mu, \tag{1.12a}$$

which implies the well known gauge transformation of the charged particle field operator

$$\Phi_\gamma{}'(x) = e^{ihx}\Phi_\gamma(x). \tag{1.12b}$$

In order to get the the gauge transformation formula (1.12b) we introduce the following transformations of $\Phi_\gamma^{(\pm)}(q)$

$$\Phi_\gamma^{(+)'}(q) = \Phi_\gamma^{(+)}(q+h); \quad \Phi_\gamma^{(+)^{+'}}(q) = \Phi_\gamma^{(+)^+}(q+h) \tag{1.13a}$$

$$\Phi_\gamma^{(-)'}(q) = \Phi_\gamma^{(-)}(q-h); \quad \Phi_\gamma^{(-)^{+'}}(q) = \Phi_\gamma^{(-)^+}(q-h). \tag{1.13b}$$

After substitution of (1.13a,b) in (1.4b) we get

$$\Phi_\gamma{}'(x) = \int \frac{d^4q}{(2\pi)^4}\Big[\Phi_\gamma^{(+)}(q+h)e^{-iqx} + \Phi_\gamma^{(-)^+}(q-h)e^{iqx}\Big] = e^{ihx}\Phi_\gamma(x) \qquad (1.14).$$

In the same way we obtain

$$\int \frac{dq_o}{2\pi}\Phi_\gamma^{(+)}(q_o+h_o,\mathbf{p}+\mathbf{h})e^{-iq_0x_0} = \int \frac{dq_o}{2\pi}\Phi_\gamma^{(+)}(q_o,\mathbf{p}+\mathbf{h})e^{-i(q_0-h_o)x_0}$$

$$= \frac{e^{-i\omega_{\mathbf{p}+\mathbf{h}}x_o}}{2\omega_{\mathbf{p}+\mathbf{h}}}a_{\mathbf{p}+\mathbf{h}\gamma}(x_o)e^{ih_ox_o} = Z^{1/2}\frac{e^{i\mathbf{p}+\mathbf{h}x}}{2\omega_{\mathbf{p}+\mathbf{h}}}\sum_\beta < 0|\Phi'_\gamma(x)|\mathbf{p}+\mathbf{h}\beta > a'_{\mathbf{p}+\mathbf{h}\beta}(x_o), \tag{1.15a}$$

where the operator

$$a'_{\mathbf{p}\gamma}(x_o) = i\sum_\beta \int d^3x < 0|\Phi'_\beta(x)|\mathbf{p}\gamma > \frac{\overleftrightarrow{\partial}}{\partial x^o}\Phi'_\beta(x) \tag{1.15b}$$

coincides with the operator (1.5b)

$$a'_{\mathbf{p}\gamma}(x_o) = a_{\mathbf{p}\gamma}(x_o). \tag{1.15c}$$

Thus the gauge transformation (1.12a,b) generates the following transformation of the $\mathcal{S}$-matrix (1.8)

$$\mathcal{S}'_{mn} =< out; \mathbf{p'_1}+\mathbf{h}\ \alpha'_1, ..., \mathbf{p'_m}+\mathbf{h}\ \alpha'_m|\mathbf{p_1}+\mathbf{h}\ \alpha_1, ..., \mathbf{p_n}+\mathbf{h}\ \alpha_n; in > \tag{1.16}$$

This expression differs from $\mathcal{S}_{mn}$ by a shift of the position of the origin in the 3D momentum space. Therefore $\mathcal{S}'_{mn} = \mathcal{S}_{mn}$ because only the relative momenta are physically meaningful. A more complicated shift of a four-momentum operator $\widehat{P}'_\mu = \widehat{P}_\mu - <0|\widehat{P}_\mu|0>$ is often used in quantum field theory concerning the so-called zero-mode problem (see for example ch. 12 of [23]).

For a neutral particle field operator $\phi_\gamma(x)$ the translation $q'_\mu = q_\mu + h_\mu$ (1.12a) has a more complicated form due to absence of the antiparticle degree of freedom. In particular, using (1.13a) we obtain

$$\phi'_\gamma(x) = \int \frac{d^4q}{(2\pi)^4}\Big[\phi_\gamma^{(+)}(q)e^{-i(q-h)x} + \phi_\gamma^{(+)^+}(q)e^{i(q-h)x}\Big] \neq e^{ihx}\phi_\gamma(x). \tag{1.17a}$$

or

$$\phi'_\gamma(x) = \int \frac{d^4q}{(2\pi)^3}\delta((q_o-h_o)^2 - (\mathbf{q}-\mathbf{h})^2 - m^2)\theta(q_o-h_o)$$

$$\Big[a'_{\mathbf{q}\gamma}(x_0)e^{-i(\omega_{\mathbf{q}-\mathbf{h}}-h_o)x_o+i(\mathbf{q}-\mathbf{h})\mathbf{x}} + a'^+{}_{\mathbf{q}\gamma}(x_o)e^{i(\omega_{\mathbf{q}-\mathbf{h}}-h_o)x_o+i(\mathbf{q}-\mathbf{h})\mathbf{x}}\Big], \tag{1.17b}$$

where

$$a'_{\mathbf{p}\gamma}(x_o) = i\sum_\beta \int d^3x <0|\phi'_\beta(x)|\mathbf{p}\gamma> \frac{\overleftrightarrow{\partial}}{\partial x^0}\phi'_\beta(x) \tag{1.18}$$

According to (1.17a,b) $\phi'_\gamma(x)$ remains Hermitian after translations $q'_\mu = q_\mu + h_\mu$. On the other hand these translations generate the nontrivial dependence of $\phi'_\gamma(x)$ on h_μ. The same dependence on the additional parameter h_μ appears in the real fields $\phi_{1,2}(x)$ constructed from the charged pion fields $\pi_\pm(x)$ after their gauge transformation (1.12a,b) $\phi'_1(x) = 1/\sqrt{2}\Big(\exp(-ihx)\pi_+(x) + \exp(ihx)\pi_+{}^+(x)\Big)$ and
$\phi'_2(x) = i/\sqrt{2}\Big(\exp(-ihx)\pi_+(x) - \exp(ihx)\pi_+{}^+(x)\Big)$. It must be noted, that a splitting of $\phi_\gamma(x)$ on the positive and the negative frequency parts $\phi_\gamma(x) = \phi_\gamma^{(+)}(x) + \phi_\gamma^{(+)^\dagger}(x)$ can be realized with arbitrary parameter α [32] as $\phi_\gamma(x) = e^{i\alpha}\phi_\gamma^{(+)}(x) + e^{-i\alpha}\phi_\gamma^{(+)^\dagger}(x)$. In our case the additional dependence of $\phi_\gamma(x)$ on h_μ is result of the condition (1.13a) which is necessary for the gauge transformations rule (1.12a,b) of the charged field operators.

Using the ortho-normalization condition for functions $f_{p-h}(x) = e^{i(p_o-h_o)x_o - i(\mathbf{p}-\mathbf{h})\mathbf{x}}$ we have

$$i\int f^*_{p'-h}(x)\frac{\overleftrightarrow{\partial}}{\partial x^0}f_{p-h}(x)d^3x = 2(p_o-h_o)(2\pi)^3\delta(\mathbf{p}'-\mathbf{p}), \tag{1.19a}$$

$$i\int f_{p'-h}(x)\frac{\overleftrightarrow{\partial}}{\partial x^o}f_{p-h}(x)d^3x = i\int f^*_{p'-h}(x)\frac{\overleftrightarrow{\partial}}{\partial x^0}f^*_{p-h}(x)d^3x = 0. \tag{1.19b}$$

It is easy to obtain

$$a'_{\mathbf{p}\gamma}(x_o) = a_{\mathbf{p}\gamma}(x_o), \tag{1.20}$$

where $a_{\mathbf{p}\gamma}(x_o) = i\sum_\beta \int d^3x < 0|\phi_\beta(x)|\mathbf{p}\gamma > \overleftrightarrow{\partial}/\partial x^0 \phi_\beta(x)$. Relation (1.20) is analogue to the relation (1.15c) for the charged fields. This means, that $\mathcal{S}$-matrix transforms according to the same relation (1.16) for the charged and neutral particles after translation in the 4D momentum space $q'_\mu = q_\mu + h_\mu$. The dependence on the dummy variables q_o and $q_o + h_o$ disappears in the $\mathcal{S}$-matrix after the appropriate integration in eq.(1.5a), eq. (1.6a) and in eq. (1.15a,b). Thus for the $\mathcal{S}$-matrix and other observables the translation of $q \equiv (q_o, \mathbf{p})$ is reduced to the 3D translations $\mathbf{p}' = \mathbf{p} + \mathbf{h}$ which does not affect these observables.

Rotation (1.1b) and dilatation (1.1c) of q_μ for the particle field operator $\Phi_\gamma(x)$ (1.11) may be performed using the rotations (1.10b) and scale transformations (1.10c) of $\Phi^{(\pm)}_\gamma(q)$ operators. In particular, rotations $q'_\mu = \Lambda_{\mu\nu}q^\nu$ generates the following transformation of the field operators in the configuration space

$$R(\Lambda): \qquad \Phi'_\gamma(x_\mu) = \Phi_\gamma(\Lambda^{-1}_{\mu\nu}x^\nu), \tag{1.21}$$

and for dilatation $q'_\mu = e^\lambda\, q_\mu$ we have

$$\mathcal{D}(\lambda): \qquad \Phi'_\gamma(x) = e^{4\lambda}\Phi_\gamma(e^{-\lambda}\, x). \tag{1.22}$$

Therefore the rotations and dilatation of $\Phi_\gamma(q)$ generate the analogical transformations of $\Phi_\gamma(x)$.

Special conformal transformation and inversion: Special conformal transformation of q_μ (1.1e) for $\Phi^{(\pm)}_\gamma(q)$ has the form

$$\Phi^{(+)'}_\gamma(q) = \Phi^{(+)}_\gamma\Big((q^I + h)^I\Big); \quad \Phi^{(+)^{+'}}_\gamma(q) = \Phi^{(+)^+}_\gamma\Big((q^I + h)^I\Big) \tag{1.23a}$$

$$\Phi^{(-)'}_\gamma(q) = \Phi^{(-)}_\gamma\Big((q^I - h)^I\Big); \quad \Phi^{(-)^{+'}}_\gamma(q) = \Phi^{(-)^+}_\gamma\Big((q^I - h)^I\Big), \tag{1.23b}$$

where the index ^{I} relates to the inversion of q_μ. According to (1.11) we get

$$\Phi'_\gamma(x) = \int \frac{d^4q}{(2\pi)^4}\Big[\Phi^{(+)}_\gamma\Big((q^I + h)^I\Big)e^{-iqx} + \Phi^{(-)^+}_\gamma\Big((q^I - h)^I\Big)e^{iqx}\Big]. \tag{1.24}$$

This formula can be essentially simplified if $\Phi_\gamma(x)$ is inversion invariant

$$\Phi^I_\gamma(x) = \int \frac{d^4q}{(2\pi)^4}\Big[\Phi^{(+)}_\gamma(q^I)e^{-iqx} + \Phi^{(-)^+}_\gamma(q^I)e^{iqx}\Big] = \Phi_\gamma(x). \tag{1.25}$$

Then after redefinition of the variables in (1.24) $\Phi^{(\pm)}_\gamma\Big((q^I \pm h)^I\Big)e^{\mp iqx} \Rightarrow \Phi^{(\pm)}_\gamma\Big((q \pm h)^I\Big)e^{\mp iqx} \Rightarrow \Phi^{(\pm)}_\gamma(q \pm h)e^{\mp iqx}$ we obtain the analogue to (1.12b) or (1.17) gauge transformation for $\Phi'_\gamma(x)$.

Arbitrary operator $\Phi^{(\pm)}_\gamma(q)$ may be divided into two parts

$$\Phi^{(\pm)}_{\gamma\;inv.}(q) = \frac{1}{2}\Big[\Phi^{(\pm)}_\gamma(q) + \Phi^{(\pm)}_\gamma(q^I)\Big] \tag{1.26a}$$

and

$$\Phi^{(\pm)}_{\gamma}{}_{ps.-inv.}(q) = \frac{1}{2}\Big[\Phi^{(\pm)}_{\gamma}(q) - \Phi^{(\pm)}_{\gamma}(q^I)\Big], \tag{1.26b}$$

where $\Phi^{(\pm)}_{\gamma}{}_{inv.}(q)$ and $\Phi^{(\pm)}_{\gamma}{}_{ps.-inv.}(q)$ denotes the inversion invariant and the inversion pseudo-invariant parts of $\Phi^{(\pm)}_{\gamma}(q)$.

The inversion invariant part of the complete field operator satisfies condition (1.25) for $\Phi^{(\pm)}_{\gamma}{}_{inv.}(q)$. An analogous condition is valid also for $\Phi^{(\pm)}_{\gamma}{}_{ps.-inv.}(q)$. Expressions (1.26a,b) enable us to simplify eq.(1.24) for the special conformal transformation of $\Phi^{(\pm)}_{\gamma}(x)$.

2. Five dimensional projection

The invariant form of the $O(2,4)$ group $\kappa_A\kappa^A = 0$ (1.2) can be represented in the five dimensional form with q_μ (1.3) variables

$$q_\mu q^\mu + M^2\frac{\kappa_-}{\kappa_+} = 0, \tag{2.1a}$$

where

$$q_\mu = \frac{\kappa_\mu}{\kappa_+}; \qquad \kappa_\pm = \frac{\kappa_5 \pm \kappa_6}{M}. \tag{2.1b}$$

It is convenient to replace two variables $\kappa_\pm$ (or κ_5, κ_6) in (2.1a) with one variable. This procedure implies a projection of the six dimensional invariant cone $\kappa_A\kappa^A = 0$ into 5D space. It exists only two 5D De Sitter spaces with the following invariant forms of the $O(2,3)$ and $O(1,4)$ rotational groups [10, 9, 18]

$$q_\mu q^\mu + q_5^2 = M^2 \qquad\qquad q_5^2 = M^2\frac{2\kappa_5}{\kappa_5+\kappa_6}, \tag{2.2a}$$

and

$$q_\mu q^\mu - q_5^2 = -M^2 \qquad\qquad q_5^2 = M^2\frac{2\kappa_6}{\kappa_5+\kappa_6}. \tag{2.2b}$$

q_μ and q_5 are real variable and they are defined in the regions $(-\infty,+\infty)$ and $[0,+\infty)$ respectively[1]. In the considered formulation q_μ and q_5 are disposed in the hyperboloids (2.2a,b). In particular, we place $0 < q^2 \le M^2$ in the hyperboloid (2.2a) and $q^2 > M^2$ can be placed only in the hyperboloid (2.2b). Thus the conformal transformations for the whole q^2 values may be performed using both hyperboloid (2.2a,b). The values of

[0]In the literature often is considered the stereographic projection of the 6D cone $\xi_A\xi^A = 0$ into 4D Minkowski space with coordinates $x_\mu = \xi_\mu\ell/(\xi_5+\xi_6)$, where at the intermediate stage are used projections on the 5D hyperboloid $\eta_\mu\eta^\mu - \eta_5^2 = -\ell^2$ (see for example ch. 13 of [24]) with $\eta_\mu = \xi_\mu\ell/\xi_5$; $\eta_5 = \xi_6\ell/\xi_5$ and $\eta_\mu = 2x_\mu/(1-x^2/\ell^2)$; $\eta_5 = \ell(1+x^2/\ell^2)/(1-x^2/\ell^2)$. Here $x^2 = \ell^2(\eta_5/\ell-1)/(\eta_5/\ell+1)$ and at first sight x_μ is not restricted by the 5D condition $\eta_\mu\eta^\mu - \eta_5^2 = -\ell^2$ like q^2 (2.2a,b) in table 1 or 2. Nevertheless, the 6D invariant form can be rewritten as $x^\mu x_\mu + \ell^2(\xi_5-\xi_6)/(\xi_5+\xi_6) = 0$ and the appropriate projection into 5D hyperboloid $x^\mu x_\mu \pm x_5^2 = \pm\ell^2$ with $x_5^2 = 2\xi_5(\ or\ \xi_6)\ell^2/(\xi_5+\xi_6)$ generates the corresponding restrictions.

q_μ and q_5 in these hyperboloids are singlevalued connected with each other via inversion $q'_\mu = -M^2 q_\mu/q^2$ (1.1d). On the 6D cone $\kappa_A\kappa^A = 0$ (1.2) inversion (1.1d) can be carried out using the reflection of the κ_6 variable

$$I(M^2): \qquad \kappa_5^I = \kappa_5, \quad \kappa_6^I = -\kappa_6; \qquad \kappa_+^I = \kappa_-, \quad \kappa_-^I = \kappa_+, \tag{2.3}$$

which generates $q_\mu^I = -M^2 q_\mu/q^2$ according to (2.1a,b). The advantage of the 6D representation (2.3) of the 4D transformation $q_\mu^I = -M^2 q_\mu/q^2$ is that it determines the transparent realization of the nonlinear 4D transformation using the simple reflection in 6D space. In particular, for $0 < q_\mu \le M^2$ on the hyperboloid $q^2 + q_5^2 = M^2$, we have ${q^2}^I = M^4/q^2 \ge M^2$ and $(-q_5^2 + M^2)^I = M^2 {\kappa_-}^I/{\kappa_+}^I = M^2/(\kappa_-/\kappa_+) = M^4/({q_5}^2 - M^2)$. Therefore, if q_μ^I belongs to (2.2b) hyperboloid ${q^2}^I + (-q_5^2 + M^2)^I = 0$, then q_μ will be placed on the other hyperboloid because ${q^2}^I + (-q_5^2 + M^2)^I = M^4/[q^2(q_5^2 - M^2)](q^2 + q_5^2 - M^2) = 0$. The distribution of the regions of the 5D hyperboloid $q^2 \pm q_5^2 = \pm M^2$ (2.2a,b) which covers the whole values of q_μ and q_5 ($-\infty < q^2 < \infty$ and $q_5^2 \ge 0$) is given in table 1.

Table 1

	I	**II**	**III**	**IV**
	$q^2 + q_5^2 = M^2$	$q^2 - q_5^2 = -M^2$	$q^2 + q_5^2 = M^2$	$q^2 - q_5^2 = -M^2$
q^2	$0 \le q^2 \le M^2$	$M^2 < q^2 < \infty$	$-\infty < q^2 < -M^2$	$-M^2 \le q^2 < 0$
q_5^2	$0 \le q_5^2 \le M^2$	$2M^2 \le q_5^2 < \infty$	$2M^2 < q_5^2 < \infty$	$0 \le q_5^2 < M^2$

In the region I momenta q_μ are singlevalued connected with q_μ in the region II via inversion $\{q_\mu\}_{II\ region} = -M^2\{q_\mu/q^2\}_{I\ region}$ and vice versa $\{q_\mu\}_{I\ region} = -M^2\{q_\mu/q^2\}_{II\ region}$. In the same way are connected the four momenta in the regions III and IV, where $q^2 < 0$. For $M \to \infty$ 5D spaces transforms into ordinary Minkowski space with the domains I and IV. For $M \to 0$ the regions II and III are remained.

The scale transformation have the different form in the different areas in table 1. In the 6D space the scale transformation $q'_\mu = e^\lambda q_\mu$ (1.1c) implies the rotation in the (6,5) plane. For $q^2 \ge 0$ rotation $\kappa_5 = M\ sh(\lambda); \kappa_6 = M\ ch(\lambda)$ generates the following transformation of q^2: $q_\mu q^\mu = -M^2(\kappa_5 - \kappa_6)/(\kappa_5 + \kappa_6) = M^2 e^{-2\lambda}$. For negative $q^2 < 0$ we take $\kappa_5 = M\ ch(\lambda)$; $\kappa_6 = M\ sh(\lambda)$ (i.e. $\kappa_+ = e^\lambda$) which gives $q_\mu q^\mu = -M^2 e^{2\lambda}$. The corresponding transformation of q_5^2 with the related λ, is given in table 2.

[1]The border point $q^2 = 0$ with $q_5^2 = M^2$ is included in the domain $q_\mu q^\mu + q_5^2 = M^2$, because it belongs to the physical spectrum of the massless particles. After inversion $q^2 = 0$ transforms into $q^2 = \infty$ of the hyperboloid $q_\mu q^\mu - q_5^2 = -M^2$ in the region II.

Table 2

rotation	$\kappa_5 = M\ sh\lambda;\ \kappa_6 = M\ ch\lambda$		$\kappa_5 = M\ ch\lambda;\ \kappa_6 = M\ sh\lambda$	
λ	$\lambda > 0$	$\lambda < 0$	$\lambda < 0$	$\lambda > 0$
	I	**II**	**III**	**IV**
hyperboloid	$q^2 + q_5^2 = M^2$	$q^2 - q_5^2 = -M^2$	$q^2 + q_5^2 = M^2$	$q^2 - q_5^2 = -M^2$
q^2	$q^2 = M^2 e^{-2\lambda}$	$q^2 = M^2 e^{-2\lambda}$	$q^2 = -M^2 e^{-2\lambda}$	$q^2 = -M^2 e^{-2\lambda}$
q_5^2	$q_5^2 = M^2(1 - e^{-2\lambda})$	$q_5^2 = M^2(1 + e^{-2\lambda})$	$q_5^2 = M^2(1 + e^{-2\lambda})$	$q_5^2 = M^2(1 - e^{-2\lambda})$

In table 2 it is shown, that the scale transformation parameter λ (or $\kappa_+ = e^{-\lambda}$) single-valued determines q_5^2 and q^2. In particular, in the region I for hyperboloid $q^2 + q_5^2 = M^2$ the scale transformation is realizable with $\lambda > 0$, which implies the compression of q^2 or q^2/M^2. In opposite to this, in the region II in the hyperboloid $q^2 - q_5^2 = -M^2$ $\lambda < 0$, i.e. the same scale transformation generates the stretching of q^2 or q^2/M^2. An analogical scale transformation can be observed for $q^2 < 0$, where in the region III dilatation generates stretching and in the region IV we have compression. In other words, projections of the 5D cone $\kappa_A\kappa^A = 0$ on the 5D hyperboloid for $q^2 > 0$ implies

$$\kappa_A\kappa^A = 0 \Longrightarrow q_\mu q^\mu + q_5^2 = M^2, \quad for\ \ 0 \le q_\mu q^\mu \le M^2\ \ with\ \ \ \lambda \ge 0 \tag{2.4a}$$

$$\kappa_A\kappa^A = 0 \Longrightarrow q_\mu q^\mu - q_5^2 = -M^2, \quad for\ \ q_\mu q^\mu > M^2\ \ with\ \ \ \lambda \le 0. \tag{2.4b}$$

Inversion ${q'}^2 = M^4/q^2$ replaces the internal points $0 \le q^2 < M^2$ (section I) by the external points $q^2 > M^2$ (section II) and vice versa. Therefore the region I my be defined as the internal region ($0 \le q_\mu q^\mu \le M^2$) and the region II ($q_\mu q^\mu > M^2$) as the external region. Thereby in order to simplify the following notation the hyperboloid $q^2 + q_5^2 = M^2$ we will denote as the "internal" surface and the hyperboloid $q^2 + q_5^2 = M^2$ we will call as the "external" domain.

By translation $q'_\mu = q_\mu + h_\mu$ the 6D cone $\kappa_A\kappa^A = 0$, as well as the 5D forms (2.2a,b) $q^2 \pm q_5^2 = \pm M^2$ are preserved. In particular, after the appropriate 6D rotations $\kappa'_\mu = \kappa_\mu + h_\mu\kappa_+$; $\kappa'_+ = \kappa_+$; $\kappa'_- = \kappa_- + 2/M^2 h_\nu\kappa^\nu + h^2/M^2\kappa_+$ we get

$$q^{2'} = q^2 + 2h_\nu q^\nu + h^2 = -M^2\frac{\kappa'_-}{\kappa_+};$$

$$q_5^{2'} = q_5^2 \pm (2h_\nu q^\nu + h^2), \tag{2.5}$$

where the sign $-$ corresponds to $q^2 + q_5^2 = M^2$ and $+$ relates to $q^2 - q_5^2 = -M^2$. Using (2.5) we get $q^{2'} \pm q_5^{2'} = q^2 \pm q_5^2$. Nevertheless, the transformation $q'_\mu = q_\mu + h_\mu$ can generate a transition from the time-like region $q^2 \ge 0$ into space-like region $q^2 < 0$ i.e. transition from the region $I\ or\ II$ into regions $III\ or\ IV$ correspondingly. Transition between the $q^2 > 0$ and $q^2 < 0$ regions is result of the transposition of κ^6 and κ^5 variables and we have used this transposition in table 2 for the scale transformations.

It must be noted, that inversion transforms the generators of the conformal group (1.10a) - (1.10d) in the following way [†2]

$$\mathcal{X}_\mu = I(M^2)\ K_\mu\ I(M^2); \quad \mathcal{M}_{\mu\nu} = I(M^2)\ \mathcal{M}_{\mu\nu}\ I(M^2);$$

$$D' = I(M^2)\ D\ I(M^2) = -D; \quad K_\mu = I(M^2)\ \mathcal{X}_\mu\ I(M^2), \tag{2.6}$$

Therefore one can perform the conformal transformations only in the "internal" regions I and III and obtain the corresponding transformations in the "external" regions II and IV using the inversion.

5D reduction of the field operators: Next we have to connect a 6D field operator $\phi^{(\pm)}(\kappa_A)$ defined in the cone $\kappa_A\kappa^A = 0$ ($A = \mu; 5, 6 \equiv 0,\ 1,\ 2,\ 3;\ 5,6$) with the 5D operators $\phi^{(\pm)}_{inr}(q, q_5)$ and $\phi^{(\pm)}_{ext}(q, q_5)$, defined in the surfaces $q^2 \pm q_5^2 = \pm M^2$ (2.2a,b). Here index $^{(\pm)}$ corresponds to positive or negative frequency, the subscripts inr or ext indicate the surfaces $q^2 + q_5^2 = M^2$ and $q^2 + q_5^2 = -M^2$ correspondingly. In particular the internal (inr) area relates to the I and III regions in table 1 and the external (ext) regions corresponds to the sections II and IV in table 1. Afterwards for the sake of simplicity we omit the spin-isospin indices γ.

According to the manifestly covariant construction of the $O(2,4)$ conformal group [16, 22], conformal transformations (1.1a)-(1.1e) are equivalent to the 6D rotations in the cone $\kappa_A\kappa^A = 0$ with the following choice of the six independent variables

$$q_\mu = \kappa_\mu/\kappa_+; \quad \kappa_+ = (\kappa_5 + \kappa_6)/M; \quad \kappa^2 = \kappa^A\kappa_A. \tag{2.7}$$

Only this choice of the variables makes independent the generators of the conformal group $O(2,4)$ on $\partial/\partial\kappa^2$. In particular, for a spinless particle these generators are

$$\mathcal{X}_\mu = i\frac{\partial}{\partial q^\mu}; \quad \mathcal{M}_{\mu\nu} = i\Big(q_\mu\frac{\partial}{\partial q^\nu} - q_\nu\frac{\partial}{\partial q^\mu}\Big);$$

$$D = i\Big(q_\mu\frac{\partial}{\partial q^\mu} - k_+\frac{\partial}{\partial k_+}\Big); \quad \mathcal{K}_\mu = 2q_\mu D - q^2\mathcal{X}_\mu. \tag{2.8}$$

Using the variables (2.7) the 6D field operator takes the form

$$\phi^{(\pm)}(\kappa_A) \equiv \phi^{(\pm)}(q_\mu, \kappa_+, \kappa^2). \tag{2.9}$$

The homogeneous over the scale variable κ_+ operator $\phi^{(\pm)}(q_\mu, \kappa_+, \kappa^2)$ may be rewritten as

$$\phi^{(\pm)}(q_\mu, \kappa_+, \kappa^2) = (\kappa_+)^d\varphi^{(\pm)}(q_\mu, \kappa^2), \tag{2.10}$$

and for the 4D physical field operator. in analogue to [16, 22] we get

$$\Phi^{(\pm)}(q_\mu) = (\kappa_+)^{-d}\mathcal{O}\phi^{(\pm)}(\kappa_A), \tag{2.11}$$

where d defines the scale dimension of the considered operator, and $\mathcal{O}$ acts on the spin-isospin variables.

In the present paper we will use other recipe of projection of the 6D cone $\kappa^2 = 0$ into 5D surfaces $q_\mu q^\mu \pm q_5^2 = \pm M^2$ and in the 4D momentum space. In particular, we will

[2] An analogical transformation can be performed using the Weyl reflection, i.e. rotation through 90^o in the $(0,5)$ plane [3].

treat the condition $\kappa^2 = 0$ as the dynamical restriction, i. e. we will require the validity of the following constraint

$$\Big(\kappa^A\kappa_A\Big)\phi^{(\pm)}(q_\mu,\kappa_+,\kappa^2) = \kappa_+^2\Big(q_\mu q^\mu + M^2\frac{\kappa_-}{\kappa_+}\Big)\phi^{(\pm)}(q_\mu,\kappa_+,\kappa^2) = 0, \qquad (2.12)$$

Projection of this equation on the 5D surfaces $q_\mu q^\mu \pm q_5^2 = \pm M^2$ gives

$$\Big(q_\mu q^\mu + q_5^2 - M^2\Big)\phi^{(\pm)}_{inr}\Big(q_\mu,\kappa_+,{\kappa_+}^2(q_\mu q^\mu + q_5^2 - M^2)\Big) = 0, \qquad (2.13a)$$

$$\Big(q_\mu q^\mu - q_5^2 + M^2\Big)\phi^{(\pm)}_{ext}\Big(q_\mu,\kappa_+,{\kappa_+}^2(q_\mu q^\mu - q_5^2 + M^2)\Big) = 0. \qquad (2.13b)$$

As it was show in table 2, the magnitude of the scale parameter $\kappa_+ = e^\lambda$ is unambiguously defined via q^2 or q_5^2 variables. Therefore we can introduce the 5D fields

$$\varphi^{(\pm)}_{inr}(q_\mu,q_5) \equiv (\kappa_+)^{-d}\mathcal{O}\phi^{(\pm)}_{inr}\Big(q_\mu,\kappa_+,{\kappa_+}^2(q_\mu q^\mu + q_5^2 - M^2)\Big), \qquad (2.14a)$$

$$\varphi^{(\pm)}_{ext}(q_\mu,q_5) \equiv (\kappa_+)^{-d}\mathcal{O}\phi^{(\pm)}_{ext}\Big(q_\mu,\kappa_+,{\kappa_+}^2(q_\mu q^\mu - q_5^2 + M^2)\Big). \qquad (2.14b)$$

Then using eq.(2.13a,b) we get

$$\Big(q_\mu q^\mu + q_5^2 - M^2\Big)\varphi^{(\pm)}_{inr}(q_\mu,q_5) = 0; \qquad (2.15a)$$

$$\Big(q_\mu q^\mu - q_5^2 + M^2\Big)\varphi^{(\pm)}_{ext}(q_\mu,q_5) = 0;. \qquad (2.15b)$$

Equations (2.15a,b) present the desired 5D projections of the 6D constraint (2.12). These relations can be treated also as the 4D equations, because the fifth momentum on $q^2 \pm q_5^2 = \pm M^2$ shell is given $q_5 = \pm\sqrt{|M^2 \mp q^2|}$.

3. 4D and 5D equations of motion in the coordinate space.

It is well known that the theories with any dimensional parameters are conformal non-invariant. Nevertheless the conformal transformations in momentum space in such theories are realizable and these transformations require the invariance of the 6D form $\kappa^2\phi(\kappa) = 0$ (2.12). Therefore we can consider this condition and the appropriate 5D projections $\Big(q_\mu q^\mu \pm {q_5}^2 \mp M^2\Big)\varphi^{(\pm)}_{inr,ext}(q_\mu,q_5) = 0$ (2.15a,b) as restrictions which can be taken into account in the 4D equation of motion

$$\Big(\frac{\partial^2}{\partial x^\mu \partial x_\mu} + m^2\Big)\Phi(x) = J(x), \qquad (3.1a)$$

where

$$J(x) = 1/(2\pi)^4\int d^4q\Big[e^{-iqx}J^{(+)}(q) + e^{iqx}J^{(-)}(q)\Big], \qquad (3.1b)$$

or

$$J(x)=\int\frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}}\Big[\frac{\partial}{\partial x_0}a_{\mathbf{p}\gamma}(x_0)e^{-ipx}+\frac{\partial}{\partial x_0}b^{+}{}_{\mathbf{p}\gamma}(x_0)e^{ipx}\Big];\quad p_o\equiv\omega_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}. \quad (3.1c)$$

In order to determine the relation between 4D eq.(3.1a) and 5D conditions (2.15a,b) we introduce the following boundary conditions over the fifth coordinate x_5

$$\Phi(x)=\Phi(x,x_5=t_5)=\varphi_{inr}(x,x_5=t_5)+\varphi_{ext}(x,x_5=t_5) \quad (3.2a)$$

$$\frac{i}{M}\frac{\partial}{\partial x_5}\Phi(x,x_5)|_{x_5=t_5}=\sum_{a=1,2}\frac{i}{M}\frac{\partial}{\partial x_5}\varphi_a(x,x_5)|_{x_5=t_5},\quad where\ \ a=1,2\equiv inr,ext \quad (3.2b)$$

and t_5 is the same boundary value of x_5 which is convenient to choose as $t_5=\tau=\sqrt{x_o^2-\mathbf{x}^2}$ or $t_5=0$, a indicates inr or ext operators.

Using (2.15a,b) we get

$$\Big(\frac{\partial^2}{\partial x^\mu\partial x_\mu}+\frac{\partial^2}{\partial x^5\partial x_5}+M^2\Big)\varphi_{inr}(x,x_5)=0 \quad (3.3a)$$

for the internal hyperboloid with the regions I, III in table 1 and

$$\Big(\frac{\partial^2}{\partial x^\mu\partial x_\mu}-\frac{\partial^2}{\partial x^5\partial x_5}-M^2\Big)\varphi_{ext}(x,x_5)=0 \quad (3.3b)$$

for the "external" regions II, IV [3].

We introduce the boundary condition for the operator $\frac{i}{M}\partial/\partial x_5\varphi(x,x_5)$ which is generated by fifth dimension

$$\frac{i}{M}\frac{\partial}{\partial x_5}\varphi(x,x_5)=\eta_a\varphi(x,x_5)+l(x,x_5);\quad a=1,2\equiv inr,ext, \quad (3.4)$$

where

$$\eta_{inr}=\sqrt{|1-m^2/M^2|};\quad \eta_{ext}=\sqrt{1+m^2/M^2}. \quad (3.5)$$

Acting with $M^2(i/M\partial/\partial x_5+\eta_a)$ on the relation (3.4) we get

[3] It must be noted, that in the usual formulation of the conformal transformations in the coordinate space one can also divide any field operator and the corresponding equations of motions into two 5D parts using inversion ${x_\mu}'=-\ell^2x_\mu/x^2$. In particular, starting from the 6D invariant form $\xi_A\xi^A=0$ with $x_\mu=\xi_\mu\ell/(\xi_5+\xi_6)$ we have $x^\mu x_\mu=-\ell^2(\xi_5-\xi_6)/(\xi_5+\xi_6)$. This condition can be projected into two 5D hyperboloid $x^\mu x_\mu\pm x_5^2=\pm\ell^2$ with $x_5^2=2\xi_5(\ or\ \xi_6)\ell^2/(\xi_5+\xi_6)$. Then we get the internal and the external 5D regions with the boundary values $x^2=0,\pm\ell^2$ as it was done for q^2,q_5^2 variables in table 1. For $\Phi(x)$ we can introduce an analogue to (3.2a,b) boundary conditions $\Phi(x)=\varphi_{inr}(x,x_5=t)+\varphi_{ext}(x,x_5=t)$. In the such constructions the operator [12] $\mathcal{M}_{\mu\nu}(x)=g_{\mu\nu}-2x_\mu x_\nu/x^2=\mathcal{M}_{\mu\nu}(1/x)$ has the properties of the metric tensor $\mathcal{M}_{\mu\nu}(x)\mathcal{M}^{\nu\sigma}(x)=\delta_\mu^\sigma$, $\mathcal{M}_{\mu\nu}(x)x^\nu=-x_\mu$ and $\partial/\partial x^\mu=1/{x'}^2\mathcal{M}_{\mu\nu}(x')\partial/\partial {x'}_\nu$. In this approach one can simplify the 5D and 4D equations in the conformal field theory in the coordinate space.

$$\Big(\frac{\partial^2}{\partial x_5 \partial x^5} + M^2 - m^2\Big)\varphi_{inr}(x,x_5) = -M^2\Big(\frac{i}{M}\frac{\partial}{\partial x_5} + \eta_{inr}\Big)l_{inr}(x,x_5) \tag{3.6a}$$

and

$$\Big(\frac{\partial^2}{\partial x_5 \partial x^5} + M^2 + m^2\Big)\varphi_{ext}(x,x_5) = -M^2\Big(\frac{i}{M}\frac{\partial}{\partial x_5} + \eta_{ext}\Big)l_{ext}(x,x_5). \tag{3.6b}$$

Combining these equations with (3.3a,b) we obtain

$$\Big(\frac{\partial^2}{\partial x_\mu \partial x^\mu} + m^2\Big)\varphi_{inr}(x,x_5) = M^2\Big(\frac{i}{M}\frac{\partial}{\partial x_5} + \eta_{inr}\Big)l_{inr}(x,x_5)$$

$$\equiv j_{inr}(x,x_5) \tag{3.7a}$$

$$\Big(\frac{\partial^2}{\partial x_\mu \partial x^\mu} + m^2\Big)\varphi_{ext}(x,x_5) = -M^2\Big(\frac{i}{M}\frac{\partial}{\partial x_5} + \eta_{ext}\Big)l_{ext}(x,x_5)$$

$$\equiv j_{ext}(x,x_5), \tag{3.7b}$$

where $j_a(x,x_5)$ are determined via $l_a(x,x_5)$.

Solutions of eq.(3.7a,b) determine the corresponding solution of the 4D equation (3.1a) with

$$J(x) \equiv J(x, x_5 = t_5) = j_{inr}(x, x_5 = t_5) + j_{ext}(x, x_5 = t_5) \tag{3.8}$$

It must be noted, that one can rewrite the boundary condition (3.4) in the integral form

$$\varphi_a(x,x_5) = e^{-iM(x_5-t_5)}\Big\{\varphi_a(x,x_5=t_5) - \frac{1}{\eta_a}l_a(x,x_5=t_5)[e^{2iM\eta_a(x_5-t_5)} - 1]$$

$$-\frac{1}{2iM\eta_a}\int_{t_5}^{x_5} dz_5 j_a(x,z_5)e^{-iM\eta_a(z_5-t_5)}\big[e^{2iM\eta_a(x_5-t_5)} - e^{2iM\eta_a(z_5-t_5)}\big]\Big\}. \tag{3.9}$$

For noninteracting particles, when $l_a(x,x_5) = 0$ and $j_a(x,x_5) = 0$, equations (3.7a,b) and the constraints (3.3a,b) and (3.4) coincide with the analoguous equations and constraints from ref. [18, 19] with the invariant form $q_\mu q^\mu + q_5^2 = M^2$ or $q_\mu q^\mu - q_5^2 = -M^2$. For the infinitely large scale parameter $M \to \infty$ this formulation [17, 18, 19] transforms into the usual 4D quantum field theory with the non restricted mass spectrum.

Consistency condition for the 5D equation of motion (3.7a,b) and the boundary conditions (3.3a,b) and (3.4).: Combining eq.(3.7a,b) and (3.6a,b) we find

$$\Big(\frac{\partial^2}{\partial x_\mu \partial x^\mu}\frac{\partial^2}{\partial x_5 \partial x^5} - \frac{\partial^2}{\partial x_5 \partial x^5}\frac{\partial^2}{\partial x_\mu \partial x^\mu}\Big)\varphi_{inr,ext}(x,x_5)$$

$$= \mp\Big(\frac{\partial^2}{\partial x_\mu \partial x^\mu} \pm \frac{\partial^2}{\partial x_5 \partial x^5} \mp M^2\Big)j_a(x,x_5) = 0. \tag{3.10a}$$

According to this relation, the 5D equations of motion (3.7a,b) are consistent with the boundary conditions (3.3a,b) and (3.4), if $j_a(x,x_5)$ and $l_a(x,x_5)$ are embedded in hyperboloid $q^2 \pm q_5^2 = \pm M^2$, i.e. in analogue to $\varphi_a(x,x_5)$ the operators $j_a(x,x_5)$ and $l_a(x,x_5)$ must satisfy the conditions

$$\Big(\frac{\partial^2}{\partial x^\mu \partial x_\mu} + \frac{\partial^2}{\partial x^5 \partial x_5} + M^2\Big) j_{inr}(x,x_5) = 0, \tag{3.10b}$$

$$\Big(\frac{\partial^2}{\partial x^\mu \partial x_\mu} - \frac{\partial^2}{\partial x^5 \partial x_5} - M^2\Big) j_{ext}(x,x_5) = 0. \tag{3.10c}$$

Using the conditions (3.3a,b) $\varphi_a(x,x_5)$ may be represented as

$$\varphi_{inr}(x,x_5) = \frac{2M}{(2\pi)^4} \int d^5q e^{-iq_5x^5} \delta(q^2 + q_5^2 - M^2)[\theta(q^2)\theta(M^2 - q^2) + \theta(-q^2)\theta(-M^2 - q^2)]$$

$$\Big[e^{-iqx}\varphi_{inr}^{(+)}(q,q_5) + e^{iqx}\varphi_{inr}^{(-)}(q,q_5)\Big], \tag{3.11a}$$

and

$$\varphi_{ext}(x,x_5) = \frac{2M}{(2\pi)^4} \int d^5q e^{-iq_5x^5} \delta(q^2 - q_5^2 + M^2)[\theta(q^2)\theta(-M^2 + q^2) + \theta(-q^2)\theta(M^2 + q^2)]$$

$$\Big[e^{-iqx}\varphi_{ext}^{(+)}(q,q_5) + e^{iqx}\varphi_{ext}^{(-)}(q,q_5)\Big]. \tag{3.11b}$$

From (3.10b,c) we get the same representation for source operator

$$j_{inr,ext}(x,x_5) = \frac{2M}{(2\pi)^4} \int d^5q e^{-iq_5x^5} \delta(q^2 \pm q_5^2 \mp M^2)[\theta(q^2)\theta(\pm M^2 \mp q^2) + \theta(-q^2)\theta(\mp M^2 \mp q^2)]$$

$$\Big[e^{-iqx}j_{inr,ext}^{(+)}(q,q_5) + e^{iqx}j_{inr,ext}^{(-)}(q,q_5)\Big]. \tag{3.12}$$

The same representation is valid for $l_{inr,ext}(x,x_5)$.

This formulation has a number of common properties with the other 5D field-theoretical approaches based on the proper time method [24, 25, 26, 27, 28], where $x_5^2 \equiv \tau = x_0^2 - \mathbf{x}^2 \equiv x_\mu x^\mu$. From this point of view the boundary conditions (3.4) can be treated as an evolution equation. On the other hand the fifth momentum q_5 is singlevalued determined via the scale parameter λ (see table 2). Therefore x_5 may be provided with the a scale interpretation if we take $\lambda^{-1} = ln(Mx_5)$. Unlike other 5D approaches [25, 29, 30, 31], in the present formulation field operators and the source operators are defined in the 5D hyperboloid.

4. 5D Lagrangian approach

The 5D operators $\varphi_{inr}(x,x_5)$ and $\varphi_{ext}(x,x_5)$ are independent because they are defined in the different domains of $q^2 \equiv q_\mu q^\mu$ and q_5^2. Therefore the sought 5D Lagrangian $\mathcal{L} \equiv \mathcal{L}(x,x_5)$ we will construct using the two sets of the independent fields $\varphi_{inr}(x,x_5)$ and $\varphi_{ext}(x,x_5)$.

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{INT} + \mathcal{L}_c, \tag{4.1a}$$

where $\mathcal{L}_0$ stands for the noninteracting part

$$\mathcal{L}_0 = \sum_{a=1,2}\Big[\frac{\partial\varphi_a(x,x_5)}{\partial x_\mu}\frac{\partial\varphi_a^+(x,x_5)}{\partial x^\mu} - m^2\varphi_a(x,x_5)\varphi_a^+(x,x_5)\Big], \tag{4.1b}$$

$a = 1,2 \equiv inr, ext$, $\mathcal{L}_{INT} \equiv \mathcal{L}_{INT}(\varphi_a, \varphi_a^+, \partial\varphi_a/\partial x_\mu, \partial{\varphi^+}_a/\partial x^\mu; \partial\varphi_a/\partial x_5, \partial\varphi_a^+/\partial x^5)$ is the interacting part of Lagrangian and $\mathcal{L}_c$ generates the constraint (3.4)

$$\mathcal{L}_c = M^2\sum_{a=1,2}|\frac{i}{M}\frac{\partial\varphi_a}{\partial x_5} - \eta_a\varphi_a - l_a(x,x_5)|^2. \tag{4.1c}$$

where $l_a(x,x_5) \equiv l_a(\varphi_a, \varphi_a^+, \partial\varphi_a/\partial x_\mu, \partial\varphi_a^+/\partial x^\mu; \partial\varphi_a/\partial x_5, \partial\varphi_a^+/\partial x^5)$ is operator from the constraint (3.4).

Next we consider action

$$\mathcal{S}(x_5) = \int d^4x\mathcal{L}(x,x_5) \tag{4.2}$$

and its variation under the conformal transformations (1.1a)-(1.1e)

$$\delta q_\mu = \delta h_\mu + \delta\Lambda_{\mu\nu}q^\nu + \delta\lambda q_\mu + (q^2\delta\hbar_\mu - 2q^\nu\delta\hbar_\nu q_\mu)/M^2 \tag{4.3a}$$

where $\delta h_\mu(\delta\hbar_\mu)$, $\delta\Lambda_{\mu\nu}q^\nu = -\delta\Lambda_{\nu\mu}q^\nu$ $\delta\lambda$ stands for the infinitesimal parameters of the corresponding transformations.

Translation $q'_\mu = q_\mu + h_\mu$ does not change x_μ. Therefore the variation of coordinates $\delta x_\mu = x'_\mu - x_\mu$, generated by variation of four momenta q_μ (4.3), includes only the rotation and the scale transformations

$$\delta x_\mu = \delta\Lambda^{-1}_{\mu\nu}x^\nu - \delta\lambda x_\mu \tag{4.3b}$$

In the considered formulation x_5 is independent variable. Therefore we take

$$\frac{\delta x_5}{\delta x_\mu} = \delta\frac{dx_5}{dx_\mu} = 0 \tag{4.4}$$

which is consistent with our choice of action (4.2).

Now we have

$$\delta\mathcal{S}(x_5) = \sum_{a=1,2}\Big\{\int d^4x\Big[\frac{\partial\mathcal{L}}{\partial\varphi_a^+(x,x_5)} - \frac{\partial}{\partial x_\mu}\Big(\frac{\partial\mathcal{L}}{\partial[\partial\varphi_a^+(x,x_5)/\partial x^\mu]}\Big)\Big]\overline{\delta}\varphi_a^+(x,x_5)$$

$$+\int d^4x \frac{d}{dx_\mu}\Big[\frac{\partial \mathcal{L}}{\partial\ [\partial\varphi_a^+(x,x_5)/\partial x^\mu]}\overline{\delta}\varphi_a^+(x,x_5)+\mathcal{L}(x,x_5)\delta x^\mu\Big]\Big\}$$

$$+\int d^4x\Big[\frac{\partial \mathcal{L}}{\partial[\partial\varphi_a^+(x,x_5)/\partial x^5]}\overline{\delta}\big(\frac{\partial}{\partial x_5}\varphi_a^+(x,x_5)\big)+\frac{d\mathcal{L}}{dx_5}\delta x^5\Big]$$

$$+\ \textit{hermitian cojugate} \tag{4.5}$$

where $\overline{\delta}$ denotes a variation of form of the corresponding expression. Substituting $d\mathcal{L}/dx_5$ in (4.5) we obtain

$$\delta\mathcal{S}(x_5)=\sum_{a=1,2}\Big\{\int d^4x\Big[\frac{\partial \mathcal{L}}{\partial\varphi_a^+(x,x_5)}-\frac{\partial}{\partial x_\mu}\Big(\frac{\partial \mathcal{L}}{\partial[\partial\varphi_a^+(x,x_5)/\partial x^\mu]}\Big)\Big]\Big[\overline{\delta}\varphi_a^+(x,x_5)+\frac{\partial\varphi_a^+(x,x_5)}{\partial x^5}\delta x_5\Big] \tag{4.6a}$$

$$+\int d^4x \frac{d}{dx_\mu}\Big[\frac{\partial \mathcal{L}}{\partial\ [\partial\varphi_a^+(x,x_5)/\partial x^\mu]}\overline{\delta}\varphi_a^+(x,x_5)+\mathcal{L}(x,x_5)\delta x^\mu\Big] \tag{4.6b}$$

$$+\int d^4x\frac{\partial \mathcal{L}}{\partial[\partial\varphi_a^+(x,x_5)/\partial x^5]}\Big[\overline{\delta}\big(\frac{\partial}{\partial x_5}\varphi_a^+(x,x_5)\big)+\frac{\partial^2\varphi_a^+(x,x_5)}{\partial {x_5}^2}\delta x_5\Big] \tag{4.6c}$$

$$+\int d^4x\frac{d}{dx_\mu}\Big[\frac{\partial \mathcal{L}}{\partial\ [\partial\varphi_a^+(x,x_5)/\partial x^5]}\frac{\partial\varphi_a^+(x,x_5)}{\partial x_\mu}\delta x_5\Big]\Big\} \tag{4.6d}$$

$$+\ \textit{hermitian cojugate}$$

In order to get $\delta\mathcal{S}(x_5)=0$ we will suppose that every term of eq.(4.6) vanishes. Then for the every term separately we obtain the following equations:

1. The first term (4.6a) represents the equation of motion for $\varphi_a^+(x,x_5)$ and $\varphi_a^+(x,x_5)$

   $$\frac{\partial \mathcal{L}}{\partial\varphi_a^+(x,x_5)}=\frac{d}{dx_\mu}\Big(\frac{\partial \mathcal{L}}{\partial[\partial\varphi_a^+(x,x_5)/\partial x^\mu]}\Big);\quad \frac{\partial \mathcal{L}}{\partial\varphi_a(x,x_5)}=\frac{d}{dx_\mu}\Big(\frac{\partial \mathcal{L}}{\partial[\partial\varphi_a(x,x_5)/\partial x^\mu]}\Big), \tag{4.7a}$$

   or

   $$\Big(\frac{\partial^2}{\partial x_\mu\partial x^\mu}+m_0^2\Big)\varphi_a(x,x_5)=\frac{\partial\mathcal{L}_{INT}}{\partial\varphi_a^+(x,x_5)}-\frac{d}{dx_\mu}\Big(\frac{\partial\mathcal{L}_{INT}}{\partial[\partial\varphi_a^+(x,x_5)/\partial x^\mu]}\Big)\equiv j_a(x,x_5) \tag{4.7b}$$

   which coincides with (3.7a,b).

2. The next term (4.6b) relates to the 4D current conservation condition

   $$\mathcal{J}^\mu(x)=\sum_{a=1,2}\mathcal{J}_a^\mu(x,x_5=t_5), \tag{4.11a}$$

   where

   $$\mathcal{J}_a^\mu(x,x_5)=\frac{\partial \mathcal{L}}{\partial\ [\partial\varphi_a^+(x,x_5)/\partial x_\mu]}\overline{\delta}\varphi_a^+(x,x_5)+\mathcal{L}(x,x_5)\delta x^\mu$$

   $$+\ \textit{hermitian cojugate}. \tag{4.7b}$$

3. Third term (4.6c) contains $\partial\varphi_a(x,x_5)/\partial x^5$. This field may be treated as independent due to fifth degrees of freedom [18, 19]. Therefore we can introduce a new kind of fields

$$\chi_a(x,x_5) = \frac{i}{M}\frac{\partial\varphi_a(x,x_5)}{\partial x_5}, \tag{4.8}$$

$\mathcal{L}_0$ does not contain these fields and they are defined via constraint Lagrangian $\mathcal{L}_c$. Using the variation principle and the independence of the fields $\chi_a(x,x_5)$ we get

$$\frac{\partial\mathcal{L}}{\partial[\partial\varphi_a^+(x,x_5)/\partial x^5]} = \frac{\partial\mathcal{L}}{\partial[\partial\varphi_a(x,x_5)/\partial x^5]} = \frac{\partial\mathcal{L}_c}{\partial[\partial\varphi_a(x,x_5)/\partial x^5]} = 0 \tag{4.9}$$

which implies

$$\chi_a - \eta_a\varphi_a - l_a(x,x_5) = -1/M^2\frac{\partial\mathcal{L}_{INT}}{\partial\chi_a^+(x,x_5)}$$
$$+\frac{\partial l_a^+}{\partial\chi_a^+(x,x_5)}\Big(\chi_a - \eta_a\varphi_a - l_a(x,x_5)\Big) + \frac{\partial l_a}{\partial\chi_a^+(x,x_5)}\Big(\chi_a^+ - \eta_a\varphi_a^+ - l_a(x,x_5)^+\Big). \tag{4.10}$$

Afterwards we restrict our formulation with such $\mathcal{L}_{INT}$ which are independent on $\partial\varphi_a/\partial x_5$ and $\partial\varphi^+/\partial x^5$. Then instead of (4.10) we get

$$\chi_a(x,x_5) - \eta_a\varphi_a(x,x_5) - l_a(x,x_5) = 0, \tag{4.11}$$

which coincides with (3.4).

Combining (3.2a,b) and (4.11) we get the connections between $l_a(x,x_5)$ and $j_a(x,x_5)$

$$j_a(x,x_5) = (-1)^{a-1}M^2\Big(\frac{i}{M}\frac{\partial}{\partial x_5} + \eta_a\Big)l_a(x,x_5). \tag{4.12}$$

which was presented in eq. (3.7a,b).

4. The fourth term (4.6d) contains the current operator

$$J_a^\mu(x,x_5) = \frac{\partial\mathcal{L}}{\partial\ [\partial\varphi_a^+(x,x_5)/\partial x^\mu]}\frac{\partial\varphi_a^+(x,x_5)}{\partial x_5}\delta x_5$$
$$+\ hermitian\ cojugate \qquad (4.13).$$

in the asymptotic region, where $j_a = 0$ and $l_a = 0$, $J_a^\mu(x,x_5)$ has the same form as the electro-magnetic current operator

$$J_a^\mu(x,x_5) = -i\eta_a\delta x_5\Big\{\frac{\partial\mathcal{L}_0}{\partial\ [\partial\varphi_a(x,x_5)/\partial x^\mu]}\varphi_a(x,x_5)$$
$$-\frac{\partial\mathcal{L}_0}{\partial\ [\partial\varphi_a^+(x,x_5)/\partial x^\mu]}\varphi_a^+(x,x_5)\Big\} \tag{4.14}$$

and $d/dx_\mu J_a^\mu(x,x_5) = 0$. For the interacting fields expression (4.13) vanishes if we require that $\delta x_5 = 0$.

Thus we have derived the equation of motion (3.7a,b), constraint (3.4) and the expression for the conserved currents (4.7b) and (4.13) using the variation principle. Combining these equations one can verify, that $d\mathcal{S}(x_5)/dx_5 = 0$, i.e. $\mathcal{S}(x_5)$ (4.2) is not dependent on x_5. In particular, using the 5D equations of motion (4.7a,b) we get $d\mathcal{S}(x_5)/dx_5 = \int d^4x d/dx_\mu \Big\{\partial\mathcal{L}/\partial[\partial\varphi_a^+(x,x_5)/\partial x^\mu]\partial\varphi_a^+(x,x_5)/\partial x_5 + \,.\,.\Big\}$ which vanishes according to fifth current conservation condition (4.13).

5. Construction of 5D Lagrangian $\mathcal{L}_{INT}$ via $l_a(x,x_5)$ (3.4)

In equations (3.7a,b) and (3.4) operators $\varphi_a(x,x_5)$, $j_a(x,x_5)$ and $l_a(x,x_5)$ are defined on the shell of the hyperboloid $q^2 \pm q_5^2 = \pm M^2$. But the product of $\varphi_a(x,x_5)$ is not on $q^2 \pm q_5^2 = \pm M^2$ shell. Therefore in order to construct some explicit representation of eq.(3.7a,b) and (3.4) it is convenient to find the simple off shell extension of $\varphi_a(x,x_5)$ and the corresponding Lagrangians. The off $q^2 \pm q_5^2 \mp M^2$ shell operator $\widetilde{\varphi}_a(x,x_5)$ can be introduced as follows

$$\varphi_a(x,x_5) = \int d^5y \widetilde{\varphi}_a(x-y,x_5-y_5)D_a(y,y_5), \tag{5.1}$$

where

$$\widetilde{\varphi}_a(x,x_5) = \frac{2M}{(2\pi)^4}\int d^5q e^{-iqx-iq_5x^5}[\theta(q^2)\theta(\pm M^2 \mp q^2) + \theta(-q^2)\theta(\mp M^2 \mp q^2)]$$
$$\Big[\varphi_{inr}^{(+)}(q,q_5) + \varphi_{inr}^{(-)}(-q,q_5)\Big], \tag{5.2a}$$

and

$$D_a(x,x_5) = \frac{1}{(2\pi)^5}\int d^5q e^{-iqx-iq_5x^5}\delta(q^2 \pm q_5^2 \mp M^2). \tag{5.3}$$

Substituting (5.2a) and (5.3) into (5.1) we obtain expressions (3.11a,b) after Fourier transforms.

Certainly, $\widetilde{\varphi}_a(x,x_5)$ is not determined via $\varphi_a(x,x_5)$ singlevalued. In particular, we can take a scale-invariant representation

$$\widetilde{\varphi}_a(x,x_5) = \frac{1}{(2\pi)^4 M}\int d^5q (Q_a)^{-4} e^{-iqx-iq_5x^5}$$
$$[\theta(q^2/(Q_a)^2)\theta(\pm M^2 \mp q^2/(Q_a)^2) + \theta(-q^2/(Q_a)^2)\theta(\mp M^2 \mp q^2/(Q_a)^2)]$$
$$\Big[\varphi_a^{(+)}(q/Q_a, q_5/Q_a) + \varphi_a^{(-)}(-q/Q_a, q_5/Q_a)\Big], \tag{5.2b}$$

where $(Q_a)^2 = (q^2 \pm q_5^2)/M^2$ and $(Q_a)^2 = 1$ on the surfaces $q^2 \pm q_5^2 = \pm M^2$. Then we obtain an equivalent representation of $\varphi_a(x,x_5)$ (3.11a,b)

$$\varphi_a(x,x_5) = \frac{2M}{(2\pi)^4}\int d^5q e^{-iqx-iq_5x^5}\delta(q^2 \pm q_5^2 \mp M^2)(Q_a)^{-4}$$
$$[\theta(q^2/(Q_a)^2)\theta(\pm M^2 \mp q^2/(Q_a)^2) + \theta(-q^2/(Q_a)^2)\theta(\mp M^2 \mp q^2/(Q_a)^2)]$$
$$\Big[\varphi_a^{(+)}(q/Q_a, q_5/Q_a) + \varphi_a^{(-)}(-q/Q_a, q_5/Q_a)\Big]. \tag{5.2c}$$

Using (5.2b) and (5.2c) we obtain the inverse to (5.1) representation

$$\widetilde{\varphi}_a(x,x_5) = \int_{0^+}^{\infty} d\alpha \varphi_a(\alpha x, \alpha x_5) \tag{5.4a}$$

which implies that

$$\frac{\delta\widetilde{\varphi}_a(x,x_5)}{\delta\varphi_a(x,x_5)} = \frac{\delta\widetilde{\varphi}_a(x,x_5)}{\delta\varphi_a(\beta x,\beta x_5)}\Big]_{\beta=1} = 1. \tag{5.4b}$$

An other operator $\mathcal{O}(x,x_5) \equiv \chi_a(x,x_5), l_a(x,x_5), j_a(x,x_5), (\mathcal{L}_a)_{INT}(x,x_5)$ may be determined via the corresponding off shell operator $\widetilde{\mathcal{O}}_a(x,x_5)$ in the same way

$$\mathcal{O}_a(x,x_5) = \int d^5y \widetilde{\mathcal{O}}_a(x-y, x_5-y_5) D_a(y,y_5) \tag{5.5a}$$

and vice versa, using the representation (5.2a,c) we get

$$\widetilde{\mathcal{O}}_a(x,x_5) = \int_{0^+}^{\infty} d\alpha \mathcal{O}_a(\alpha x, \alpha x_5). \tag{5.5b}$$

The straightforward generalization of equations (3.7a,b) is not available for $\widetilde{\varphi}_a(x,x_5)$ (5.2b) and (5.4a). In particular, if we take in eq.(3.7a,b) $x_\mu = \alpha y_\mu$ and $x_5 = \alpha y_5$, then after integration over α we get

$$\frac{\partial^2}{\partial y_\mu \partial y^\mu}\int_{0^+}^{\infty}\frac{d\alpha}{\alpha^2}\varphi_a(\alpha y, \alpha y_5) + m^2\widetilde{\varphi}_a(y,y_5) = \widetilde{j}_a(y,y_5). \tag{5.6}$$

On the other hand starting from the equations of motion

$$\Big[\frac{\partial^2}{\partial x_\mu \partial x^\mu} + m^2\Big]\widetilde{\varphi}_a(x-y, x_5-y_5) = \widetilde{j}_a(x-y, x_5-y_5), \tag{5.7}$$

we obtain the equations of motion (3.7a,b) using integration over y, y_5 variables according to eq. (5.1). Certainly, in eq.(5.6) and in eq.(5.7) the different expressions of $\widetilde{\varphi}_a$ are assumed.

In the same way as eq.(4.7a,b) with the constraint (4.11) we can derive equation of motion (5.7) and the off shell constraint for the off $q^2 \pm q_5^2 \mp M^2$ shell Lagrangian

$$\widetilde{\mathcal{L}}_a = (\widetilde{\mathcal{L}}_a)_0 + (\widetilde{\mathcal{L}}_a)_{INT} + (\widetilde{\mathcal{L}}_a)_c, \tag{5.8a}$$

where $(\widetilde{\mathcal{L}}_a)_0$ stands for the noninteracting part, $(\widetilde{\mathcal{L}}_a)_{INT}$ is the interaction part and $(\widetilde{\mathcal{L}}_a)_c$ generate the constraint

$$\widetilde{\chi}_a(x,x_5) - \eta_a\widetilde{\varphi}_a(x,x_5) - \widetilde{l}_a(x,x_5) = 0 \qquad (5.9a)$$

and have the form

$$(\widetilde{\mathcal{L}})_c = M^2 \sum_{a=1,2} |\frac{i}{M}\frac{\partial\widetilde{\varphi}_a}{\partial x_5} - \eta_a\widetilde{\varphi}_a - \widetilde{l}_a(x,x_5)|^2. \qquad (5.9b)$$

Thus relation (5.1) enables us to obtain the straightforward off shell representations of eq.(3.7a,b) and the constraint (3.4) using the same as (4.1a,b,c) Lagrangian but with off $q^2 \pm q_5^2 \mp M^2$ shell operators $\widetilde{\varphi}_a$. Now we consider some example of $\widetilde{l}_a(x,x_5)$ and the corresponding interaction Lagrangians:

<u>φ^4 **model:**</u> The simplest $\widetilde{l}_a$ which does not dependent on $\widetilde{\chi}_a(x,x_5) \equiv i/M\ \partial\widetilde{\varphi}_a(x,x_5)/\partial x_5$ is

$$\widetilde{l}_a = g_a\widetilde{\varphi}_a^2, \qquad (5.10)$$

Using the constraint (5.9a) we get

$$\frac{i}{M}\frac{\partial\widetilde{\varphi}_a(x,x_5)}{\partial x_5} = \eta_a\ \widetilde{\varphi}_a(x,x_5) + g_a\widetilde{\varphi}_a^2. \qquad (5.11)$$

$$\widetilde{j}_a(x,x_5) = \partial\widetilde{\mathcal{L}}_{INT}/\partial\widetilde{\varphi}_a^+(x,x_5) = (-1)^{a-1}M^2\Big(\frac{i}{M}\frac{\partial}{\partial x_5} + \eta_a\Big)\widetilde{l}_a(x,x_5)$$
$$= (-1)^{a-1}M^2\Big(3g_a\eta_a\widetilde{\varphi}_a^2 + 2g_a^2\widetilde{\varphi}^3\Big). \qquad (5.12)$$

The corresponding equation of motion can be derived using the following Lagrangians

$$\widetilde{\mathcal{L}} = \frac{1}{2}\sum_{a=1,2}\Big[\frac{\partial\widetilde{\varphi}_a}{\partial x_\mu}\frac{\partial\widetilde{\varphi}_a}{\partial x^\mu} - m_a^2\widetilde{\varphi}_a^2\Big] + M^2\sum_{a=1,2}|\ \widetilde{\chi}_a - \eta_a\widetilde{\varphi}_a - g_a\widetilde{\varphi}_a^2\ |^2 + \widetilde{\mathcal{L}}_{INT}, \qquad (5.13)$$

where

$$(\widetilde{\mathcal{L}}_a)_{INT}(x,x_5) = (-1)^{a-1}M^2\Big(g_a\eta_a\widetilde{\varphi}_a^3 + \frac{g_a^2}{2}\widetilde{\varphi}_a^4\Big) \qquad (5.14)$$

Lagrangian (5.13) has the following attractive properties

I. The considered model is renormalizable, because $\widetilde{\mathcal{L}}_{INT}$ and $\widetilde{\mathcal{L}}$ contains $\widetilde{\varphi}_a$ in the third and in the fourth power.

II. $(\widetilde{\mathcal{L}}_{inr})_{INT}$ $(a=1)$ and $(\widetilde{\mathcal{L}}_{ext})_{INT}$ $(a=2)$ have the opposite sign.

III. The Lagrangian (5.14) has a local minimum at $-2\eta_a/g_a$ and a local maximum at $-\frac{3}{2}\eta_a/g_a$.

Nonlinear σ model: In this case we have pi-meson fields $\pi^{\pm}$, π^0 instead of φ. We choose l_a depending on the auxiliary fields χ

$$\widetilde{l}_a^{\alpha} = \frac{1}{4f_\pi^2}(\widetilde{\chi}_a^{\gamma}\widetilde{\chi}_a^{\gamma})\widetilde{\pi}_a^{\alpha} \equiv \frac{1}{4f_\pi^2}\widetilde{\chi}^2\widetilde{\pi}_a^{\alpha}, \tag{5.15}$$

where we have used well known isospin redefinition of the pi-meson fields $\pi^{\pm} \equiv 1/2(\pi^1 \pm i\pi^2)$; $\pi^0 \equiv \pi^3$; $\alpha, \beta, \gamma = 1, 2, 3$, $f_\pi = 93MeV$ is the pi-meson decay constant and Lagrangian is choosing in the form

$$\widetilde{\mathcal{L}} = \sum_{a=1,2}\Big(\frac{1}{2}\frac{\partial}{\partial x_\mu}\widetilde{\pi}_a^{\alpha}\frac{\partial}{\partial x_\mu}\widetilde{\pi}_a^{\alpha} + M^2\Big[\widetilde{\chi}_a^{\alpha} - \widetilde{\pi}_a^{\alpha} - \frac{1}{4f_\pi^2}\widetilde{\chi}_a^2\widetilde{\pi}_a^{\alpha}\Big]^2\Big) + \widetilde{\mathcal{L}}_{chir} + \widetilde{\mathcal{L}}_{INT}, \tag{5.16}$$

where the second term generates the constraint between the auxiliary field $\widetilde{\chi}_a^{\alpha}(x, x_5) = i/M\partial\widetilde{\pi}_a^{\alpha}(x, x_5)/\partial x_5$ and the π meson field $\widetilde{\pi}_a^{\alpha}$

$$\widetilde{\chi}_a^{\alpha} - \widetilde{\pi}_a^{\alpha} - \frac{1}{4f_\pi^2}\widetilde{\chi}_a^2\widetilde{\pi}_a^{\alpha} = 0. \tag{5.17a}$$

This constraint coincides with the relation between the π meson field and the interpolating field in the nonlinear σ-model [33, 13]

$$\widetilde{\pi}_a^{\alpha} = \frac{1}{1 + \frac{\widetilde{\chi}_a^2}{4f_\pi^2}}\widetilde{\chi}_a^{\alpha}, \tag{5.17b}$$

Third term of (5.16) $\widetilde{\mathcal{L}}_{chir}$ reproduces the constraint between pi-meson fields and the auxiliary σ-meson fields

$$\widetilde{\pi}_a^2 + \widetilde{\sigma}_a^2 = f_\pi^2 \tag{5.18}$$

and correspondingly

$$(\widetilde{\mathcal{L}}_a)_{chir}(x, x_5) = \Big(\widetilde{\pi}_a^2 + \widetilde{\sigma}_a^2 - f_\pi^2\Big)^2. \tag{5.19}$$

In the usual σ model the chiral symmetry is weakly broken with the additional Lagrangian $\mathcal{L}' = -f_\pi m_\pi^2\sigma$. In the considered model the chiral symmetry breaking terms arise in $\widetilde{\mathcal{L}}_{INT}$. This Lagrangian may be constructed using the source operator $\widetilde{j}_a^{\alpha}$ which is defined via operator (5.15)

$$\widetilde{j}_a^{\alpha}(x, x_5) = \partial(\widetilde{\mathcal{L}}_a)_{INT}/\partial\widetilde{\pi}_a^{\alpha}(x, x_5) = (-1)^{a-1}M^2\Big(\frac{i}{M}\frac{\partial}{\partial x_5} + 1\Big)\widetilde{l}_a^{\alpha}(x, x_5)$$

$$= (-1)^{a-1}M^2\frac{(f_\pi + \widetilde{\sigma}_a)}{\widetilde{\sigma}_a}\Big[1 + f_\pi\frac{(3f_\pi - \widetilde{\sigma}_a)}{(f_\pi - \widetilde{\sigma}_a)^2}\Big]\widetilde{\pi}_a^{\alpha}. \tag{5.20}$$

The corresponding Lagrangian is

$$\widetilde{\mathcal{L}}_{INT} = -\sum_{a=1,2}(-1)^{a-1}M^2\Big(f_\pi\widetilde{\sigma}_a + \frac{1}{2}\widetilde{\sigma}_a^2 + f_\pi\frac{(f_\pi + \widetilde{\sigma}_a)^2}{(f_\pi - \widetilde{\sigma}_a)}\Big) \tag{5.21a}$$

which after expansion in π_a^2 power series takes the form

$$\widetilde{\mathcal{L}}_{INT} = -\sum_{a=1,2}(-1)^{a-1}M^2\Big(-\frac{9}{2}f_\pi^2 + 8\frac{f_\pi^4}{\widetilde{\pi}_a^2} - \widetilde{\pi}_a^2 - \frac{1}{4f_\pi^2}\widetilde{\pi}_a^4 - ...\Big) \qquad (5.21b)$$

Thus $\widetilde{\mathcal{L}}_{INT}$ for the $ext = a \equiv 2$ induces the real π meson mass term if M is fixed as

$$M = \frac{m_\pi}{\sqrt{2}}. \qquad (5.22)$$

and for the internal π meson field $\widetilde{\pi}_{a=1}$ in (5.21b) appear only negative m_π^2, i.e. $\widetilde{\pi}_{a=1}$ remines to be massless.

The suggested 5D Lagrangian allows us to reproduce exactly the nonlinear σ Lagrangian in the region with $q^2 > M^2 = m_\pi^2/2$. Moreover the considered model we have reproduced explicitely the chiral symmetry breaking term in the σ models $-m_\pi^2 f_\pi \widetilde{\sigma}_a$ in (5.21a) together with the other chiral symmetry breaking terms. In the limit $m_\pi \to 0$ (i. e. $M^2 \to 0$,) the above Lagrangian transforms into the free Lagrangian for the massless pion. Note that the chiral symmetry breaking mechanism allowed us to fix the scale parameter of the conformal transformation group via the pion mass.

6. Models with the gauge transformations

Gauge transformation in the 4D and 5D coordinate space.

Using the 6D rotations we can perform translations $q'_\mu = q_\mu - eA_\mu(q)$ (or $q'_\mu = q_\mu - eA_\mu(q, q_5)$) always supposing that the 6D cone $\kappa_A\kappa^A = 0$ and its 5D projections $q^2 \pm {q_5}^2 = \pm M^2$ are invariant. In particular, in analogue to eq.(2.5), translations $q'_\mu = q_\mu - eA_\mu(q)$ imply the following transformations of the 6D variables

$$\kappa'_\mu = \kappa_\mu - ea_\mu(\kappa_A)\kappa_+; \qquad \kappa'_+ = \kappa_+;$$

$$\kappa'_- = \kappa_- - e/M^2\Big(a_\nu(\kappa_A)\kappa^\nu + \kappa^\nu a_\nu(\kappa_A)\Big) + e^2/M^2\kappa_+ a_\nu(\kappa_A)a^\nu(\kappa_A).$$

After these transformations we get $q^{2'} = q^2 - e\Big(A_\nu(q, q_5)q^\nu + q^\nu A_\nu(q, q_5)\Big) + e^2A_\nu(q, q_5)A^\nu(q, q_5)$ and $q_5^{2'} = q_5^2 \mp e\Big(A_\nu(q, q_5)q^\nu + q^\nu A_\nu(q, q_5)\Big) + e^2A_\nu(q, q_5)A^\nu(q, q_5)$, where the sign $-$ corresponds to $q^2 + q_5^2 = M^2$ and $+$ relates to $q^2 - q_5^2 = -M^2$. $A_\nu(q, q_5)$ is constructed by $a_\nu(\kappa_A)$ according to eq.(2.11). As a result of these gauge transformations we have $q^{2'} \pm q_5^{2'} = q^2 \pm q_5^2$.

In order to derive equations of motions using the gauge transformation in the 4D and 5D coordinate space, we consider first the off $q^2 \pm {q_5}^2 = \pm M^2$ shell 5D equations

$$\Big(\frac{\partial^2}{\partial x_\mu \partial x^\mu} + m^2\Big)\widetilde{\varphi}_a(x, x_5) = \widetilde{j}_a(x, x_5), \qquad (6.1a)$$

where the source operator $\widetilde{j}_a(x,x_5)$ is defined via the 4D gauge transformations $q'_\mu = q_\mu - eA_\mu(q,q_5)$

$$\widetilde{j}_a(x,x_5) = \Big(ie\frac{\partial}{\partial x_\mu}\widetilde{A}^a_\mu(x,x_5) + ie\widetilde{A}^a_\mu(x,x_5)\frac{\partial}{\partial x_\mu} - e^2\widetilde{A}^a_\mu(x,x_5)\widetilde{A}^{\mu\ a}(x,x_5)\Big)\widetilde{\varphi}_a(x,x_5). \quad (6.1b)$$

Using relations (5.1) and (5.5a) we can embed eq.(6.1a,b) on $q^2 \pm {q_5}^2 = \pm M^2$ shell

$$\Big(\frac{\partial^2}{\partial x_\mu \partial x^\mu} + m^2\Big)\varphi_a(x,x_5) = j_a(x,x_5). \quad (6.1c)$$

Next we can determine $\widetilde{l}_a(x,x_5)$ via $j_a(x,x_5)$

$$\widetilde{l}_a(q,q_5) = (-1)^{a-1}\frac{\widetilde{j}_a(q,q_5)}{M(q_5 + M\eta_a)} \quad (6.2a)$$

and reproduce the corresponding constraint

$$\frac{i}{M}\frac{\partial}{\partial x_5}\widetilde{\varphi}_a(x,x_5) = \eta_a\widetilde{\varphi}_a(x,x_5) + \widetilde{l}_a(x,x_5). \quad (6.2b)$$

Afterwards one can build corresponding off $q^2 \pm {q_5}^2 = \pm M^2$ shell Lagrangian

$$\widetilde{\mathcal{L}} = \sum_{a=1,2}\Big[\frac{\partial\widetilde{\varphi}_a(x,x_5)}{\partial x_\mu}\frac{\partial\widetilde{\varphi}^+_a(x,x_5)}{\partial x^\mu} - m^2\widetilde{\varphi}_a(x,x_5)\widetilde{\varphi}^+_a(x,x_5) + M^2\Big|\frac{i}{M}\frac{\partial\widetilde{\varphi}_a}{\partial x_5} - \eta_a\widetilde{\varphi}_a - \widetilde{l}_a(x,x_5)\Big|^2\Big]$$

$$+ \sum_{a=1,2}\Big[-ie\widetilde{\varphi}_a(x,x_5)\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\widetilde{\varphi}^+_a(x,x_5)\widetilde{A}^{a\mu}(x,x_5) + e^2\widetilde{A}_{a\mu}(x,x_5)\widetilde{A}^{a\mu}(x,x_5)\widetilde{\varphi}_a(x,x_5)\widetilde{\varphi}^+_a(x,x_5)\Big]. \quad (6.2d)$$

This Lagrangian reproduces the equation of motion (6.1a) and the constraint (6.2b) using the 4D gauge transformation $q'_\mu = q_\mu - e(\widetilde{A}_a)_\mu(q,q_5)$ in the off $q^2 \pm {q_5}^2 = \pm M^2$ shell regions. On $q^2 \pm {q_5}^2 = \pm M^2$ shell we obtain eq.(6.1c).

On the other hand the 4D gauge transformations $q'_\mu = q_\mu - eA_\mu(q)$ generates the ordinary 4D gauge equations

$$\Big(\frac{\partial^2}{\partial x_\mu \partial x^\mu} + m^2\Big)\Phi(x) = J(x) = \Big(ie\frac{\partial}{\partial x_\mu}A_\mu(x) + ieA_\mu(x)\frac{\partial}{\partial x_\mu} - e^2A_\mu(x)A^\mu(x)\Big)\Phi(x), \quad (6.3a)$$

This equation can be divided into two equations using eq.(3.8) and eq.(3.2a)

$$J(x) = j_{inr}(x,t_5) + j_{ext}(x,t_5), \qquad \Phi(x) = \phi_{inr}(x,t_5) + \phi_{ext}(x,t_5) \quad (6.3b)$$

with arbitrary M and

$$\Big(\frac{\partial^2}{\partial x_\mu \partial x^\mu}+m^2\Big)\phi_a(x,x_5) = j_a(x,x_5) = \Big(ie\frac{\partial}{\partial x_\mu}A_\mu(x)+ieA_\mu(x)\frac{\partial}{\partial x_\mu}-e^2A_\mu(x)A^\mu(x)\Big)\phi_a(x,x_5). \tag{6.3c}$$

The equations (6.3c) are the 5D representation of the 4D equations (6.3a), where $x_5 = t_5$ and the 4D gauge transformation is assumed. The source operator (6.1b) differs from the source operator (6.3b). Thus the 4D and the 5D gauge transformations generates the different 4D equations (6.3a).

Using the exact form of $j_a(x, x_5)$ (6.3c) we can construct

$$l_a(q,q_5) = (-1)^{a-1}\frac{j_a(q,q_5)}{M(q_5+M\eta_a)} \tag{6.4a}$$

and reproduce the corresponding constraint

$$\frac{i}{M}\frac{\partial}{\partial x_5}\phi_a(x,x_5) = \eta_a\phi_a(x,x_5) + l_a(x,x_5) \tag{6.4c}$$

This enables us to construct the corresponding on $q^2 \pm {q_5}^2 = \pm M^2$ shell Lagrangian.

$$\mathcal{L} = \sum_{a=1,2}\Big[\frac{\partial\phi_a(x,x_5)}{\partial x_\mu}\frac{\partial\phi_a^+(x,x_5)}{\partial x^\mu} - m^2\phi_a(x,x_5)\phi_a^+(x,x_5) + M^2\Big|\frac{i}{M}\frac{\partial\phi_a}{\partial x_5} - \eta_a\phi_a - l_a(x,x_5)\Big|^2\Big]$$

$$+\sum_{a=1,2}\Big[-ie\phi_a(x,x_5)\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\phi_a^+(x,x_5)A^\mu(x) + e^2A_\mu(x)A^\mu(x)\phi_a(x,x_5)\phi_a^+(x,x_5)\Big]. \tag{6.4d}$$

This Lagrangian reproduces the 4D equation of motion (6.3a) with the conditions (6.3b). But Lagrangian (6.4d) differs from Lagrangian (6.2d) due to A_μ and $j_a(x, x_5)$ operators.

In the same way we can redefine the equation of motion for the fermion field operator

$$\Big(i\gamma_\mu\frac{\partial}{\partial x_\mu} - m_{el}\Big)\Psi(x) = e\gamma_\mu A^\mu(x)\Psi(x) \equiv J(x) \tag{6.5}$$

using the appropriate 5D equation

$$\Big(i\gamma_\mu\frac{\partial}{\partial x_\mu} - m_{el}\Big)\widetilde{\psi}_a(x,x_5) = \widetilde{j}_a(x,x_5), \tag{6.6a}$$

where $\widetilde{j}_a(x, x_5)$ is constructed from $J(x)$ (6.5) and generally it does not coincide with the pure 5D gauge source operator $e\gamma_\mu\widetilde{A}^\mu_a(x, x_5)\widetilde{\psi}_a(x, x_5)$. The fermion field $\widetilde{\psi}_a(x, x_5)$ satisfies the following constraints

$$\Big(\frac{i}{M}\frac{\partial}{\partial x_5} - \eta_a\Big)\widetilde{\psi}_a(x,x_5) = \widetilde{l}_a(x,x_5), \tag{6.6b}$$

$$(-1)^{a-1}M^2\Big(\frac{i}{M}\frac{\partial}{\partial x_5}+\eta_a\Big)\widetilde{l}_a(x,x_5)=-\widetilde{j}_a(x,x_5). \qquad (6.6c)$$

In the equations (6.1)-(6.6c) M is a free parameter. If $M=0$ or $M\to\pm\infty$ we obtain the usual 4D quantum field formulation without splitting into the "inr" and "out" regions in the momentum space.

Gauge $SU(2)\times U(1)$ theory (see for example [34]) can by formulated in the 5D form using the following off $q^2\pm {q_5}^2=\pm M^2$ shell Lagrangian

$$\widetilde{\mathcal{L}}(x,x_5)=\sum_{a=1,2}(\widetilde{\mathcal{L}}_a)_V(x,x_5)+(\widetilde{\mathcal{L}}_a)_{sk}(x,x_5)+(\widetilde{\mathcal{L}}_a)_F(x,x_5), \qquad (6.7)$$

where $(\widetilde{\mathcal{L}}_a)_V$ contains the vector Yang-Mills fields $(\widetilde{A}_a)^\alpha_\mu(x,x_5)$ and Abelian fields $(\widetilde{B}_a)_\mu(x,x_5)$

$$(\widetilde{\mathcal{L}}_a)_V=-\frac{1}{4}(F_a)^\alpha_{\mu\nu}(F_a)^{\mu\nu\alpha}-\frac{1}{4}(G_a)_{\mu\nu}(G_a)^{\mu\nu}$$
$$-M^2\Big|\frac{i}{M}\frac{\partial}{\partial x_5}(\widetilde{A}_a)^\alpha_\mu-(\widetilde{A}_a)^\alpha_\mu-(\widetilde{l}^A_a)^\alpha_\mu\Big|^2-M^2\Big|\frac{i}{M}\frac{\partial}{\partial x_5}(\widetilde{B}_a)_\mu-(\widetilde{B}_a)_\mu-(\widetilde{l}^B_a)_\mu\Big|^2, \qquad (6.8a)$$

where

$$(F_a)^\alpha_{\mu\nu}=\frac{\partial}{\partial x^\mu}(\widetilde{A}_a)^\alpha_\nu-\frac{\partial}{\partial x^\nu}(\widetilde{A}_a)^\alpha_\mu+g\varepsilon^{\alpha\beta\gamma}(\widetilde{A}_a)^\beta_\mu(\widetilde{A}_a)^\gamma_\nu, \qquad (6.8b)$$

$$(G_a)_{\mu\nu}=\frac{\partial}{\partial x^\mu}(\widetilde{B}_a)_\nu-\frac{\partial}{\partial x^\nu}(\widetilde{B}_a)_\mu. \qquad (6.8c)$$

We define $(\widetilde{l}^A_a)^\alpha_\mu$, $(\widetilde{l}^B_a)_\mu$ via $(\widetilde{j}^A_a)^\alpha_\mu$, $(\widetilde{j}^B_a)_\mu$ in the same way as in eq.(6.2a,b). The interacting parts of the gauge fields $(\widetilde{B}_a)_\mu$ and $(\widetilde{A}_a)^\alpha_\mu$ are contained in the fermion and in the scalar terms $(\widetilde{\mathcal{L}}_a)_F$ and $(\widetilde{\mathcal{L}}_a)_{sk}$.

The scalar part of Lagrangian (6.7) is

$$(\widetilde{\mathcal{L}}_a)_{sk}=\Big(\mathcal{D}^\mu_a\widetilde{\Phi}_a\Big)^\dagger\Big(\mathcal{D}_{a\mu}\widetilde{\Phi}_a\Big)-M^2\Big|\frac{i}{M}\frac{\partial}{\partial x_5}(\widetilde{\Phi}_a)-\widetilde{\Phi}_a-\widetilde{l}^\Phi_a\Big|^2+(\widetilde{\mathcal{L}}_a)_{INT}, \qquad (6.9a)$$

where $(\widetilde{\mathcal{L}}_a)_{INT}$ contains the self-interaction term of the scalar particle and

$$\mathcal{D}_{a\mu}\widetilde{\Phi}_a=\Big(\frac{\partial}{\partial x^\mu}-ig\frac{\tau^\alpha}{2}(\widetilde{A}_a)^\alpha_\mu-ig'(\widetilde{B}_a)_\mu\Big)\widetilde{\Phi}_a. \qquad (6.9b)$$

Unlike the standard $SU(2)\times U(1)$ theory, we start from the massless $\Phi_a(x,x_5)$ field and the Higgs mechanism we will reproduce using

$$\widetilde{l}^\Phi_a(x,x_5)=-\frac{f}{M}\widetilde{\Phi}_a(x,x_5)\Big(\widetilde{\Phi}^*_a(x,x_5)\widetilde{\Phi}_a(x,x_5)\Big)^{1/2}. \qquad (6.10)$$

In the unitary gauge

$$\widetilde{\Phi}_a(x,x_5) = \mathcal{U}\Big(\widetilde{\zeta}_a(x,x_5)\Big)\begin{pmatrix} 0 \\ \widetilde{\phi}_a(x,x_5) \end{pmatrix}, \tag{6.11a}$$

where

$$\mathcal{U}\Big(\widetilde{\zeta}_a(x,x_5)\Big) = exp\Big(i\widetilde{\zeta}_a(x,x_5)/v\Big). \tag{6.11b}$$

We will assume, that $\widetilde{\zeta}_a$ is independent on x_5

$$\frac{\partial}{\partial x_5}\widetilde{\zeta}_a(x,x_5) = 0. \tag{6.12}$$

Afterwards $\widetilde{l}_a$ (6.10) takes a form

$$\widetilde{l}^{\Phi}_a(x,x_5) = -\frac{f}{M}\mathcal{U}\Big(\widetilde{\zeta}_a(x,x_5)\Big)\begin{pmatrix} 0 \\ \widetilde{\phi}_a(x,x_5) \end{pmatrix}\sqrt{\Big(\widetilde{\phi}_a(x,x_5)\Big)^2}, \tag{6.13}$$

and for $\frac{i}{M}\partial/\partial x_5\widetilde{\phi}_a$ we get

$$\frac{i}{M}\frac{\partial}{\partial x_5}(\widetilde{\phi}_a) = \widetilde{\phi}_a - \frac{f}{M}\widetilde{\phi}_a\sqrt{(\widetilde{\phi}_a)^2}. \tag{6.14}$$

Next for $\widetilde{j}_a$ and for $(\widetilde{\mathcal{L}}_a)_{INT}$ we have

$$\widetilde{j}_a = (-1)^{a-1}\mathcal{U}(\widetilde{\zeta}_a)M^2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big(-3fM\widetilde{\phi}_a\sqrt{(\widetilde{\phi}_a)^2} + 2f^2\widetilde{\phi}^3_a\Big) \tag{6.15}$$

and

$$(\widetilde{\mathcal{L}}_a)_{INT} = (-1)^{a-1}\Big(-fM\widetilde{\phi}^2_a\sqrt{(\widetilde{\phi}_a)^2} + \frac{f^2}{2}\widetilde{\phi}^4_a\Big) \tag{6.16}$$

For the positive f Lagrangian $\widetilde{\mathcal{L}}_{a=1\equiv inr}$ is similar to the self-interaction potential

$$V(\Phi) = -\mu^2\Phi^2 + \lambda\Phi^4. \tag{6.17}$$

In particular $\widetilde{\mathcal{L}}_1$ has zero at $\widetilde{\phi}_1 = 0$ and $\widetilde{\phi}_1 = \pm 2M/f$ and $\widetilde{\mathcal{L}}_1$ has minima at $\widetilde{\phi}_1 = \pm 3M/2f$. It is important to note, that $\widetilde{\mathcal{L}}_{a=2\equiv ext} = -\widetilde{\mathcal{L}}_{a=1\equiv inr}$. Therefore in $\widetilde{\mathcal{L}}_2$ only the negative m^2 may be appear. For the spontaneous symmetry breaking case we define

$$< 0|\phi_a(x,x_5)|0 >= \frac{1}{2}\begin{pmatrix} 0 \\ \mathrm{v}_a \end{pmatrix}, \tag{6.18}$$

where $\mathrm{v}_a = \mathrm{v}$ for $a = 1 \equiv inr$ and $\mathrm{v}_a = 0$ for $a = 2 \equiv ext$. Afterwards we get

$$\widetilde{\phi}_1(x,x_5) = \frac{1}{2}\begin{pmatrix} 0 \\ \mathrm{v} + \widetilde{\phi}'(x,x_5) \end{pmatrix}; \qquad \widetilde{\phi}_2(x,x_5) = \frac{1}{2}\begin{pmatrix} 0 \\ \widetilde{\phi}'(x,x_5) \end{pmatrix}. \tag{6.19}$$

Therefore Lagrangian (6.16) takes the form

$$(\widetilde{\mathcal{L}}_a)_{INT} = (-1)^{a-1}\Big(-\frac{fM}{4}(\widetilde{\phi}'_a + \mathrm{v}_a)^2\sqrt{(\widetilde{\phi}'_a + \mathrm{v}_a)^2} + \frac{f^2}{16}(\widetilde{\phi}'_a + \mathrm{v}_a)^4\Big). \qquad (6.20)$$

Thus instead of the mass term in the usual 4D self interacting potential (6.17) (see for instance ch. 8 and 11 of [34]) in the 5D Lagrangian (6.20) arise the following terms

$$\mu^2\Big(\frac{1}{\sqrt{2}}(\Phi' + v)\Big)^2 \Longrightarrow \frac{fM}{4}\Big(\widetilde{\phi}'_{inr} + \mathrm{v}\Big)^2\sqrt{\Big(\widetilde{\phi}'_{inr} + \mathrm{v}\Big)^2} - \frac{fM}{4}(\widetilde{\phi}'_{ext})^2\sqrt{(\widetilde{\phi}'_{ext})^2}, \qquad (6.21)$$

which determine the mass spectrum.

The effective Lagrangian (6.21) has minima at $\widetilde{\phi}_1 = \pm 3M/f$. Therefore $\widetilde{\mathcal{L}}_{a=1}$ does not contain the linear terms in $\widetilde{\phi}_1$, i.e. $\widetilde{j}_1$ does not include the constant terms. If we take $v = \sqrt{2}\mathrm{v} = 3\sqrt{2}M/f$, then in the considered model all expressions for the fermion masses were reproduced, i.e. $m_k = f_k v/\sqrt{2}$, $k = el, u, d$ and the W, Z meson masses remain be the same $m_Z = m_W/cos\theta_W$. But, for the Higgs boson mass from Lagrangian (6.21) we get $m^2_{higgs} = 9/8M^2$. Thus the principal difference between the suggested 5D formulation and the standard $SU(2)\times U(1)$ gauge field theory consist in the symmetry breaking terms (6.21). The scale parameter M is defined via the mass of the Higgs boson and it indicates the border $q^2 = \pm M^2$, where the interaction of the scalar fields change the sign.

7. Conclusion

This paper is devoted to the conformal transformations of the interacted quantum fields in the momentum space. The key point of the present formulation is the invariance of the 6D cone $\kappa_\mu\kappa^\mu + \kappa_5^2 - \kappa_6^2 = 0$ (1.2) under a conformal transformation of any field operator. Therefore the 5D forms $q^2 \pm q_5^2 = \pm M^2$ (2.2a,b), arising via projection of this 6D cone into 4D momentum space $q_\mu = \frac{\kappa_\mu}{\kappa_+}$; $\kappa_\pm = (\kappa_5 \pm \kappa_6)/M$; $\mu = 0, 1, 2, 3$ (1.3), are also invariant. This invariance was taken into account by derivation of a 5D equation of motion and the corresponding 5D Lagrangians for a interacting massive particles. The suggested system of the 5D equations of motion contains separately the one-dimensional equations over the fifth coordinate x_5. These one dimensional equations have the form of the evaluation equations if $x_5^2 = x_o^2 - \mathbf{x}^2$. and the evolution operator of these equations $l_a(x, x_5)$ is connected with the particle source operator $j_a(x, x_5) = (-1)^{a-1}M^2(i/M\partial/\partial x_5 + M)l_a(x, x_5)$.

Special attention was given to the inversion of momenta $q'_\mu = -M^2 q_\mu/q^2$. In particular, the whole definition area of q^2 and q_5^2 was divided into four internal and external regions (see table 1 and 2) which are connected via inversion. The 4D field operator $\Phi(x)$ was constructed via 5D operators $\varphi_{inr}(x, x_5)$ and $\varphi_{ext}(x, x_5)$ using the boundary condition $\Phi(x) = \varphi_{inr}(x, x_5 = t_5) + \varphi_{ext}(x, x_5 = t_5)$ (3.2a).

The important parameter of the considered 5D formulation is the scale constant M which is necessary by inversion $q'_\mu = -M^2 q_\mu/q^2$ and by determination of momenta q_μ through the 6D variables $\kappa_\mu, \kappa_\pm$. In models, where the operators $l_a(x, x_5)$ from the fifth

dimension boundary condition are determined via the source operators $j_a(x,x_5)$ (see for example (6.2a,b) and (6.4a)) M is not fixed. In the theories with the spontaneous breaking symmetry M is defined via the proper mass of these theories. For instance, for the nonlinear σ-model $M = m_\pi/\sqrt{2}$ and $m_{higgs} = 3/2\sqrt{2}M$ in the 5D formulation of the standard $SU(2) \times U(1)$ theory. In the present 5D nonlinear σ-model the $\lambda\sigma$ and other chiral symmetry broken terms are explicitely reproduced. In the domain $q^2 \geq m_\pi^2/2$ this 5D model coincides with the Weinberg nonlinear σ model [33, 13] and in the internal domain $(0 \leq q^2 < m_\pi^2/2)$ the π meson field is massless. It must be noted, that in the considered 5D formulation the interaction Lagrangians and the corresponding source operators change their sign at the border $q^2 = \pm M^2$.

I am sincerely grateful to V.G.Kadyshevsky for numerous constructive and fruitful discussions.

References

[1] H. A. Kastrup, Phys. Rev.**150**(1966) 1183;

[2] L. Castell, Nucl. Phys. **B4**(1967) 343.

[3] P. Budinich and R. Raczka, Found. Phys.**23**(1993) 599.

[4] P. Budinich, Found. Phys. **32**(2002) 1347.

[5] A.O.Barut and R. Raczka, Theory of Group Representations and Applications;PWN,Warszawa, 1977.

[6] H. Bacry, Ann. Ins. H. Pouncare, **49**(1988) 245; H. Bacry, Localizabelity and Space in Quantum Physics; Lect. Notes in Phys. **308** (Springer, Berlin Heidelberg, 1988).

[7] V.M. Braun, G. P. Korchemsky and D. Müller, Prog. Part. Nucl. Phys. **51** (2003) 311.

[8] E. S. Fradkin and M.Y. Palchik, Conformal Quantum Field Theory in D-Dimensions, in: Mathematics and its Applications V.376 Kluwer,Dordrecht, Netherlands, New York, 1996; E. S. Fradkin and M.Y. Palchik, Phys.Rep.**300** (1998) 1.

[9] I. T. Todorov, Conformal Description of Spinning Particles; Springer, New York, 1986; Todorov I. T., Minchev M.C. and Petkova V.B. // Conformal covariance in quantum field theory; (Scuola Normale Superiore, Pisa, 1978.

[10] J. Backers, J. Harnad, M. Perroud and P. Winternitz, J. Math. Phys.**19**(1978) 2126.

[11] B.G. Konopelchenko,Sov. J. Elem. Part. and At. Nucl. (in Russian) **11**(1977) 135.

[12] S. Ferrara, R. Gatto and A.F. Grilo, Ann. Phys. **76** (1973) 161; S. Ferrara, R. Gatto and A.F. Grilo,in Scale and Conformal Symmetry in Hadron Physics (ed. R. Gatto), Wiley, New York, 1974;

[13] V. De Alfaro, S. Fubini, G. Furlan and C. Rosseti, Currents in Hadron Physics; (North-Holland, Amsterdam) 1973.

[14] L. Castell, Nuovo Cim.**A46**(1966) 1; L. Castell, Nuovo Cim.**A49**(1967) 285.

[15] P.A.M. Dirac, Ann. Math. **37** (1936) 429.

[16] G. Mack and A. Salam, Ann. Phys.**53**(1969) 174.

[17] V. G. Kadyshevsky, J.Exp. Theor. Phys. (in Russian)**41**(1961) 1885.

[18] V. G. Kadyshevsky, Sov.J. Elem. Part. and At. Nucl. (in Russian) **11**(1980) 5; Preprint JINR,(in Russian) 2-84-753.

[19] V. G. Kadyshevsky and M. D. Mateev, Nuovo Cim.**87A**(1985)324; M. V. Chizhov, A. D. Donkov, R. M. Ibadov, V. G. Kadyshevsky and M. D. Mateev, Nuovo Cim.**87A**(1985)351 and 373.

[20] W. Heisenberg, Ann. Phys. (Leipzig)**5**(1938) 20; H. P. Dürr and W. Heisenberg, Z. Natur. **16a**(1961) 726.

[21] M. A. Markov, Suppl. Prog. Theor. Phys. Commamemory Issue for 30-th Annivever-sary of Meson Theory by Dr. A. Yukawa (1965) 865; J.Exp. Theor. Phys. (in Russian) **51**(1966) 878.

[22] G. Mack and I. T. Todorov, Phys. Rev.**D8**(1973) 1764.

[23] J. R. Bjorken and S. D. Drell. Relativistic Quantum Fields; New York, McGrew-Hill, 1963.

[24] C. Itzykson and J. B. Zuber. Quantum Field Theory; New York, McGrew-Hill, 1980.

[25] J. L. Fanchi, Found. Phys.**22**(1993)487.

[26] M. C. Land, Found. Phys.**27**(1997) 19.

[27] R. Kubo, Nuovo Cim.**A85**(1985)293.

[28] D. M. Greenberger, J. Math. Phys.**11**(1970)2329 and 2341; ibid. **15**(1974)395 and 406.

[29] H.S. Snyder, Phys. Rev.**71** (1947) 384; ibid **72** (1947) 68.

[30] C.N. Yang, Phys. Rev. **72** (1947) 874L.

[31] M.R. Hamilton and G. Sandri, Phys. Rev. **D8** (1973)1788.

[32] S. Weinberg, The Quantum Theory of Fields; Cambridge, University Press, 1995.

[33] S. Weinberg, Phys. Rev.**D2**(1970)674; ibid **177**(1969)2604.

[34] T.-P. Cheng and L.-F. Li, Gauge Theory of Elementary Particles; (Clarendo Press, Oxford) 1984.